

A THEORY OF HIERARCHICAL CONSEQUENCE AND CONDITIONALS

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Abstract

We introduce \mathcal{A} -ranked preferential structures and combine them with an accessibility relation. \mathcal{A} -ranked preferential structures are intermediate between simple preferential structures and ranked structures. The additional accessibility relation allows us to consider only parts of the overall \mathcal{A} -ranked structure. This framework allows us to formalize contrary to duty obligations, and other pictures where we have a hierarchy of situations, and maybe not all are accessible to all possible worlds. Representation results are proved.

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1 Introduction

1.1 Description of the problem

This paper, like all papers about nonmonotonic logics, is about formalization of (an aspect of) common sense reasoning.

We often see a hierarchy of situations, e.g.:

- (1) it is better to prevent an accident than to help the victims,
- (2) it is better to prove a difficult theorem than to prove an easy lemma,
- (3) it is best not to steal, but if we have stolen, we should return the stolen object to its legal owner, etc.

On the other hand, it is sometimes impossible to achieve the best objective.

We might have seen the accident happen from far away, so we were unable to interfere in time to prevent it, but we can still run to the scene and help the victims.

We might have seen friends last night and had a drink too many, so today's headaches will not allow us to do serious work, but we can still prove a little lemma.

We might have needed a hammer to smash the windows of a car involved in an accident, so we stole it from a building site, but will return it afterwards.

We see in all cases:

- a hierarchy of situations
- not all situations are possible or accessible for an agent.

In addition, we often have implicitly a “normality” relation:

Normally, we should help the victims, but there might be situations where not: This would expose ourselves to a very big danger, or this would involve neglecting another, even more important task (we are supervisor in a nuclear power plant . . .), etc.

Thus, in all “normal” situations where an accident seems imminent, we should try to prevent it. If this is impossible, in all “normal” situations, we should help the victims, etc.

We combine these three ideas

- (1) normality,
- (2) hierarchy,
- (3) accessibility

in the present paper.

Note that it might be well possible to give each situation a numerical value and decide by this value what is right to do - but humans do not seem to think this way, and we want to formalize human common sense reasoning.

Before we begin the formal part, we elaborate above situations with more examples.

- We might have the overall intention to advance computer science.

So we apply for the job of head of department of computer science at Stanford, and promise every tenured scientist his own laptop.

Unfortunately, we do not get the job, but become head of computer science department at the local community college. The college does not have research as priority, but we can still do our best to achieve our overall intention, by, say buying good books for the library, or buy computers for those still active in research, etc.

So, it is reasonable to say that, even if we failed in the best possible situation - it was not accessible to us - we still succeeded in another situation, so we achieved the overall goal.

- The converse is also possible, where better solutions become possible, as is illustrated by the following example.

The daughter and her husband say to have the overall intention to start a family life with a house of their own, and children.

Suppose the mother now asks her daughter: You have been married now for two years, how come you are not pregnant?

Daughter - we cannot afford a baby now, we had to take a huge mortgage to buy our house and we both have to work.

Mother - *I* shall pay off your mortgage. Get on with it!

In this case, what was formerly inaccessible, is now accessible, and if the daughter was serious about her intentions - the mother can begin to look for baby carriages.

Note that we do not distinguish here how the situations change, whether by our own doing, or by someone else's doing, or by some events not controlled by anyone.

- Consider the following hierarchy of obligations making fences as unobtrusive as possible, involving contrary to duty obligations.

- (1) You should have no fence (main duty).

- (2) If this is impossible (e.g. you have a dog which might invade neighbours' property), it should be less than 3 feet high (contrary to duty, but second best choice).
- (3) If this is impossible too (e.g. your dog might jump over it), it should be white (even more contrary to duty, but still better than nothing).
- (4) If all is impossible, you should get the neighbours' consent (etc.).

1.2 Outline of the solution

The last example can be modelled as follows ($\mu(x)$ is the minimal models of x) :

Layer 1: $\mu(True)$: all best models have no fence.

Layer 2: $\mu(fence)$: all best models with a fence are less than 3 ft. high.

Layer 3: $\mu(fence \text{ and more than 3 ft. high})$: all best models with a tall fence have a white fence.

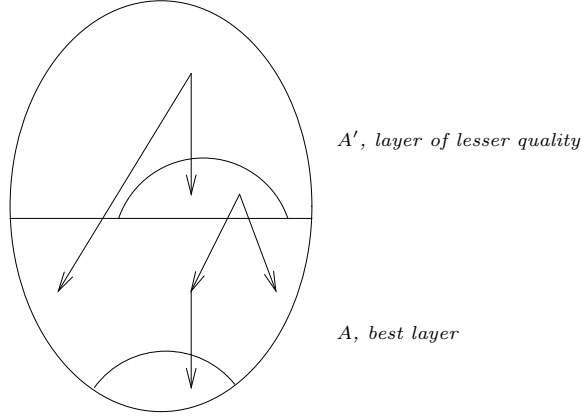
Layer 4: $\mu(fence \text{ and non-white and } \geq 3 \text{ ft})$: in all best models with a non-white fence taller than 3 feet, you have permission

Layer 5: all the rest

This will be modelled by a corresponding \mathcal{A} -structure.

In summary:

- (1) We have a hierarchy of situations, where one group (e.g. preventing accidents) is strictly better than another group (e.g. helping victims).
- (2) Within each group, preferences are not so clear (first help person A, or person B, first call ambulance, etc.?).
- (3) We have a subset of situations which are attainable, this can be modelled by an accessibility relation which tells us which situations are possible or can be reached.



*Each layer behaves inside like any preferential structure.
Amongst each other, layers behave like ranked structures.*

\mathcal{A} -ranked structure

Diagram 1.1

We combine all three ideas, consider what we call \mathcal{A} -ranked structures, structures which are organized in levels A_1, A_2, A_3 , etc., where all elements of A_1 are better than any element of A_2 - this is basically rankedness $>$, and where inside each A_i we have an arbitrary relation of preference. Thus, an \mathcal{A} -ranked structure is between a simple preferential structure and a fully ranked structure.

See Diagram 1.1.

Remark: It is not at all necessary that the rankedness relation between the different layers and the relation inside the layers express the same concept. For instance, rankedness may express deontic preference, whereas the inside relation expresses normality or some usualness.

In addition, we have an accessibility relation R , which tells us which situations are reachable.

It is perhaps easiest to motivate the precise choice of modelling by layered (or contrary to duty) obligations.

For any point t , let $R(t) := \{s : tRs\}$, the set of R -reachable points from t . Given a preferential structure $\mathcal{X} := \langle X, < \rangle$, we can relativize \mathcal{X} by considering only those points in X , which are reachable from t .

Let $X' \subseteq X$, and $\mu(X')$ the minimal points of X' , we will now consider $\mu(X') \cap R(t)$ - attention, not: $\mu(X' \cap R(t))$! This choice is motivated by the following: norms are universal, and do not depend on one's situation t .

If \mathcal{X} describes a simple obligation, then we are obliged to Y iff $\mu(X') \cap R(t) \neq \emptyset$, and $\mu(X') \cap R(t) \subseteq Y$. The first clause excludes obligations to the unattainable. We can write this as follows, supposing that X' is the set of models of ϕ' , and Y is the set of models of ψ :

$$m \models \phi' > \psi.$$

Thus, we put the usual consequence relation \vdash into the object language as $>$, and relativize to the attainable (from m).

If an \mathcal{A} -ranked structure has two or more layers, then we are, if possible, obliged to fulfill the lower obligation, e.g. prevent an accident, but if this is impossible, we are obliged to fulfill the upper obligation, e.g. help the victims, etc.

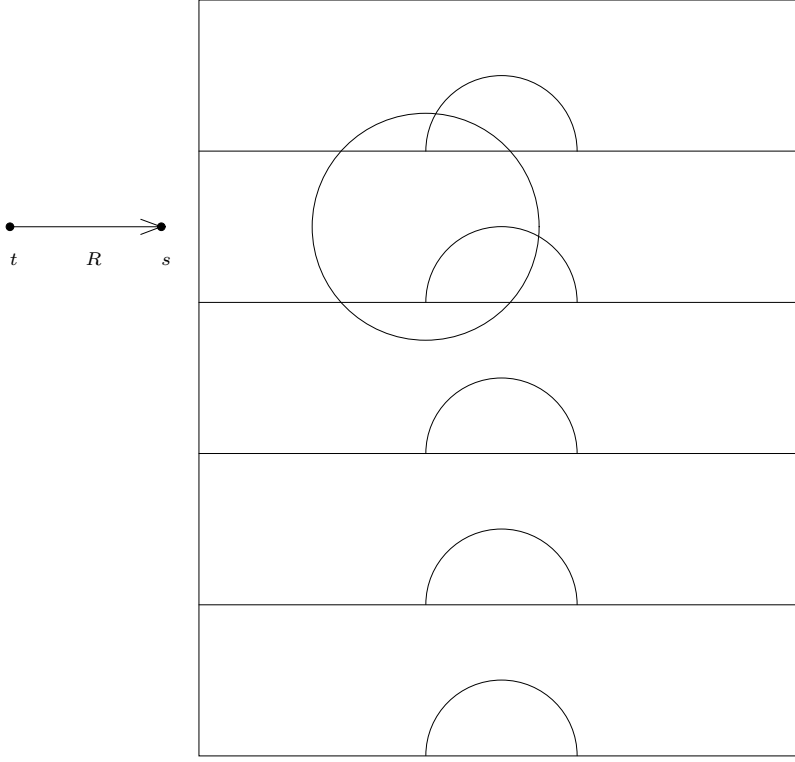
See Diagram 1.2.

Diagram 1.2

The overall structure is visible from t

Only the inside of the circle is visible from s

Half-circles are the sets of minimal elements of layers



\mathcal{A} - ranked structure and accessibility

Let now, for simplicity, \mathbf{B} be a subset of the union of all layers \mathbf{A} , and let \mathbf{B} be the set of models of β . This can be done, as the individual subset can be found by considering $A \cap \mathbf{B}$, and call the whole structure $\langle \mathcal{A}, \mathbf{B} \rangle$.

Then we say that m satisfies $\langle \mathcal{A}, \mathbf{B} \rangle$ iff in the lowest layer \mathbf{A} where $\mu(A) \cap R(m) \neq \emptyset$ $\mu(A) \cap R(m) \subseteq \mathbf{B}$.

When we want a terminology closer to usual conditionals, we may write e.g. $(A_1 > B_1; A_2 > B_2; \dots)$ expressing that the best is A_1 , and then B_1 should hold, the second best is A_2 , then B_2 should hold, etc. (The B_i are just $A_i \cap \mathbf{B}$.) See Diagram 4.1.

1.3 Historical remarks

- (1) In an abstract consideration of desirable properties a logic might have, [Gab85] examined rules a nonmonotonic consequence relation \sim should satisfy:

$$(1.1) \text{ (REF) } \Delta, \alpha \sim \alpha,$$

$$(1.2) \text{ (CUM) } \Delta \sim \alpha \Rightarrow (\Delta \sim \beta \Leftrightarrow \Delta, \alpha \sim \beta).$$

Preferential structures themselves were introduced as abstractions of Circumscription independently in [Sho87b] and [BS85]. A precise definition of these structures is given below in Definition 3.5.

Both, the semantic and the syntactic, approaches were connected in [KLM90], where a representation theorem was proved, showing that the (stronger than *Gabbay's*) system P corresponds to “smooth” preferential structures. System P consists of

$$(1.1) \text{ (AND) } \phi \sim \psi, \phi \sim \psi' \Rightarrow \phi \sim \psi \wedge \psi',$$

$$(1.2) \text{ (OR) } \phi \sim \psi, \phi' \sim \psi \Rightarrow \phi \vee \phi' \sim \psi,$$

$$(1.3) \text{ (LLE) } \vdash \phi \leftrightarrow \phi' \Rightarrow (\phi \sim \psi \Leftrightarrow \phi' \sim \psi),$$

$$(1.4) \text{ (RW) } \phi \sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow \phi \sim \psi',$$

$$(1.5) \text{ (SC) } \vdash \phi \rightarrow \phi' \Rightarrow \phi \sim \phi',$$

$$(1.6) \text{ (CUM) } \phi \sim \psi \Rightarrow (\phi \sim \psi' \Leftrightarrow \phi \wedge \psi \sim \psi').$$

where \vdash is classical provability.

Details can be found in Definition 3.4.

- (2) Ranked preferential structures were introduced in [LM92], see Definition 3.8. On the logical side, they correspond to above system P , plus the additional axiom:

$$\text{(RatM) } \phi \sim \psi, \phi \not\sim \neg\psi' \Rightarrow \phi \wedge \psi' \sim \psi.$$

- (3) Accessibility relations in possible worlds semantics go back (at least) to Kripke’s semantics for modal logics.

1.4 Formal modelling and summary of results

We started with an investigation of “best fulfillment” of abstract requirements, and contrary to duty obligations.

It soon became evident that semi-ranked preferential structures give a natural semantics to contrary to duty obligations, just as simple preferential structures give a natural semantics to simple obligations - the latter goes back to Hansson [Han69].

A semi-ranked - or \mathcal{A} -ranked preferential structure, as we will call them later, as they are based on a system of sets \mathcal{A} - has a finite number of layers, which amongst them are totally ordered by a ranking, but the internal ordering is just any (binary) relation. It thus has stronger properties than a simple preferential structure, but not as strong ones as a (totally) ranked structure.

The idea is to put the (cases of the) strongest obligation at the bottom, and the weaker ones more towards the top. Then, fulfillment of a strong obligation makes the whole obligation automatically satisfied, and the weaker ones are forgotten.

Beyond giving a natural semantics to contrary to duty obligations, semi-ranked structures seem very useful for other questions of knowledge representation. For instance, any blackbird might seem a more normal bird than any penguin, but we might not be so sure within each set of birds.

Thus, this generalization of preferential semantics seems very natural and welcome.

The second point of this paper is to make some, but not necessarily all, situations accessible to each point of departure. Thus, if we imagine agent a to be at point p , some fulfillments of the obligation, which are reachable to agent a' from point p' might just be impossible to reach for him. Thus, we introduce a second relation, of accessibility in the intuitive sense, denoting situations which can be reached. If this relation is transitive, then we have restrictions on the set of reachable situations: if p is accessible from p' , and p can access situation s , then so can p' , but not necessarily the other way round.

On the formal side, we characterize:

- (1) \mathcal{A} -ranked structures,
- (2) satisfaction of an \mathcal{A} -ranked conditional once an accessibility relation between the points p, p' , etc. is given.

For the convenience of the reader, we now state the main formal results of this paper - together with the more unusual definitions.

On (1):

Let \mathbf{A} be a fixed set, and \mathcal{A} a finite, totally ordered (by $<$) disjoint cover by non-empty subsets of \mathbf{A} .

For $x \in \mathbf{A}$, let $rg(x)$ the unique $A \in \mathcal{A}$ such that $x \in A$, so $rg(x) < rg(y)$ is defined in the natural way.

A preferential structure $\langle \mathcal{X}, \prec \rangle$ (\mathcal{X} a set of pairs $\langle x, i \rangle$) is called \mathcal{A} -ranked iff for all x, x' $rg(x) < rg(x')$ implies $\langle x, i \rangle \prec \langle x', i' \rangle$ for all $\langle x, i \rangle, \langle x', i' \rangle \in \mathcal{X}$. See Definition 3.5 for the definition of preferential structures, and Diagram 1.1 for an illustration.

We then have:

Let \sim be a logic for \mathcal{L} . Set $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, and $\overline{\overline{T}} := \{\phi : T \sim \phi\}$. where \mathcal{M} is a preferential structure.

- (1) Then there is a (transitive) definability preserving classical preferential model \mathcal{M} s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff

(LLE), (CCL), (SC), (PR) hold for all $T, T' \subseteq \mathcal{L}$.

- (2) The structure can be chosen smooth, iff, in addition

(CUM) holds.

- (3) The structure can be chosen \mathcal{A} -ranked, iff, in addition

(\mathcal{A} -min) $T \not\vdash \neg\alpha_i$ and $T \not\vdash \neg\alpha_j$, $i < j$ implies $\overline{\overline{T}} \vdash \neg\alpha_j$

holds.

See Definition 3.6 for the logic defined by a preferential structure, Definition 3.4 for the logical conditions, Definition 3.7 for smoothness.

On (2)

Given a transitive accessibility relation R , $R(m) := \{x : mRx\}$.

Given \mathcal{A} as above, let $\mathbf{B} \subseteq \mathbf{A}$ be the set of “good” points in \mathbf{A} , and set $\mathcal{C} := \langle \mathcal{A}, \mathbf{B} \rangle$.

We define:

- (1) $\mu(\mathcal{A}) := \bigcup \{\mu(A_i) : i \in I\}$

(warning: this is NOT $\mu(\mathbf{A})$)

- (2) $\mathcal{A}_m := R(m) \cap \mathbf{A}$,
- (3) $\mu(\mathcal{A}_m) := \bigcup \{ \mu(A_i) \cap R(m) : i \in I \}$
- (3a) $\nu(\mathcal{A}_m) := \mu(\mu(\mathcal{A}_m))$
- (thus $\nu(\mathcal{A}_m) = \{ a \in \mathbf{A} : \exists A \in \mathcal{A} (a \in \mu(A), a \in R(m), \text{ and } \neg \exists a' (\exists A' \in \mathcal{A} (a' \in \mu(A'), a' \in R(m), a' \prec a)) \}$).
- (4) $m \models \mathcal{C} :\leftrightarrow \nu(\mathcal{A}_m) \subseteq \mathbf{B}$.

See Diagram 4.1

Then the following hold:

Let $m, m' \in M$, $A, A' \in \mathcal{A}$, \mathbf{A} be the set of models of α .

- (1) $m \models \Box \neg \alpha, m R m' \Rightarrow m' \models \Box \neg \alpha$
- (2) $m R m', \nu(\mathcal{A}_m) \cap A \neq \emptyset, \nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset, \Rightarrow A \leq A'$ (in the ranking)
- (3) $m R m', \nu(\mathcal{A}_m) \cap A \neq \emptyset, \nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset, m \models \mathcal{C}, m' \not\models \mathcal{C}, \Rightarrow A < A'$

Conversely, these conditions suffice to construct an accessibility relation between M and \mathbf{A} satisfying them, so they are sound and complete.

1.5 Overview

We next point out some connections with other domains of artificial intelligence and computer science.

We then put our work in perspective with a summary of logical and semantical conditions for nonmonotonic and related logics, and present basic definitions for preferential structures.

Next, we will give special definitions for our framework.

We then start the main formal part, and prove representation results for \mathcal{A} -ranked structures, first for the general case, then for the smooth case. The general case needs more work, as we have to do a (minor) modification of the not \mathcal{A} -ranked case. The smooth case is easy, we simply have to append a small construction. Both proofs are given in full detail, in order to make the text self-contained.

Finally, we characterize changes due to restricted accessibility.

2 Connections with other concepts

2.1 Hierarchical conditionals and programs

Our situation is now very similar to a sequence of computer program instructions:

if A_1 then do B_1 ;

else if A_2 then do B_2 ;

else if A_3 then do B_3 ;

where we can see the B_i as subroutines.

We can deepen this analogy in two directions:

(1) connect it to Update

(2) put an imperative touch to it.

In both cases, we differentiate between different degrees of fulfillment of \mathcal{C} : the lower the level is which is fulfilled, the better.

(1) We can consider all threads of reachability which lead to a model m where $m \models \mathcal{C}$. Then we take as best threads those which lead to the best fulfillment of \mathcal{C} . So degree of fulfillment gives the order by which we should do the update. (This is then not update in the sense that we choose the most normal developments, but rather we actively decide for the most desirable ones.) We will not pursue this line any further here, but leave it for future research.

(2): We introduce an imperative operator, say!! means that one should fulfill \mathcal{C} as best as possible by suitable choices. We will elaborate this now.

First, we can easily compare the degree of satisfaction of \mathcal{C} of two models:

Definition 2.1

Let $m, m' \models \mathcal{C}$, and define $m < m' :\Rightarrow \mu(\mu(\mathcal{A}_m) \cup \mu(\mathcal{A}_{m'})) \cap \mu(\mathcal{A}_{m'}) = \emptyset$. (μ is, as usual, relative to some fixed \leq_t .)

For two sets of models, X, X' , the situation does not seem so easy. So suppose that $X, X' \models \mathcal{C}$. First, we have to decide how to compare this, we do by the maximum: $X < X'$ iff the worst satisfaction of all $x \in X$ is better than the worst satisfaction in X' . More precisely, we look at all $\gamma(\mathcal{C})$ for all $x \in X$, take the maximum (which exists, as \mathcal{A} is finite), and then compare the maxima for X and for X' .

Suppose now that there are points where we can make decisions ("free will"), let m be such a point. We introduce a new relation D , and let mDm' iff we can decide to go from m to m' . The relation D expresses this possibility - it is our definition of "free will".

Definition 2.2

Consider now some formula ϕ , and define

$m \models !\phi :\Rightarrow D(m) \cap M(\phi) < D(m) \cap M(\neg\phi)$

(as defined in Definition 2.1).

2.2 Connection with Theory Revision

In particular, the situation of contrary to duty obligations (see Section ??) shows an intuitive similarity to revision. You have the duty not to have a fence. If this is impossible (read: inconsistent), then it should be white. So the duty is revised.

But there is also a formal analogy: As is well known, AGM revision (with fixed left hand side K) corresponds to a ranked order of models, where models of K have lowest rank (or: distance 0 from K -models). The structures we consider (\mathcal{A} -rankings) are partially ranked, i.e. there is only a partial ranked preference, inside the layers, nothing is said about the ordering. This partial ranking is natural, as we have only a limited number of cases to consider.

But we use the revision order (based on K , so it really is a \leq_K relation) differently: We do not revise K , but use only the order to choose the first layer which has non-empty intersection with the set of possible cases. Still, the spirit (and formal apparatus) of revision is there, just used somewhat differently. The K -relation expresses here deontic quality, and if the best situation is impossible, we choose the second best, etc.

Theory revision with variable K is expressed by a distance between models (see [LMS01]), where $K * \phi$ is defined by the set of ϕ models which have minimal distance from the set of K models.

We can now generalize our idea of layered structure to a partial distance as follows: For instance, $d(K, A)$ is defined, $d(K, B)$ too, and we know that all A models with minimal distance to K have smaller distance than the B models with minimal distance to K . But we do NOT know a precise distance for other A models, we can sometimes compare, but not always. We may also know that all A models are closer to K than any B model is, but for a and a' , both A models, we might not know if one or the other is closer to K , or if they have the same distance.

2.3 Definitions for our framework

3 Representation results for \mathcal{A} -ranked structures

3.1 Basic definitions for preferential structures

Definition 3.1

We use \mathcal{P} to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \rightarrow \bigcup\{X_i : i \in I\}, \forall i \in I. g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X , and V the set-theoretic universe we work in - the class of all sets. Given a set of pairs \mathcal{X} , and a set X , we denote by $\mathcal{X} \upharpoonright X := \{ \langle x, i \rangle \in \mathcal{X} : x \in X \}$. When the context is clear, we will sometime simply write X for $\mathcal{X} \upharpoonright X$.

$A \subseteq B$ will denote that A is a subset of B or equal to B , and $A \subset B$ that A is a proper subset of B , likewise for $A \supseteq B$ and $A \supset B$.

Given some fixed set U we work in, and $X \subseteq U$, then $C(X) := U - X$.

If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X , we say that \mathcal{Y} satisfies

- (\cap) iff it is closed under finite intersections,
- (\bigcap) iff it is closed under arbitrary intersections,
- (\cup) iff it is closed under finite unions,

(\bigcup) iff it is closed under arbitrary unions,

(C) iff it is closed under complementation.

We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.

We make ample and tacit use of the Axiom of Choice.

Definition 3.2

\prec^* will denote the transitive closure of the relation \prec . If a relation $<$, \prec , or similar is given, $a \perp b$ will express that a and b are $< -$ (or $\prec -$) incomparable - context will tell. Given any relation $<$, \leq will stand for $<$ or $=$, conversely, given \leq , $<$ will stand for \leq , but not $=$, similarly for \prec etc.

Definition 3.3

We work here in a classical propositional language \mathcal{L} , a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ , ψ , etc., theories T , S , etc.

$v(\mathcal{L})$ will be the set of propositional variables of \mathcal{L} .

$M_{\mathcal{L}}$ will be the set of (classical) models of \mathcal{L} , $M(T)$ or M_T is the set of models of T , likewise $M(\phi)$ for a formula ϕ .

$\mathbf{D}_{\mathcal{L}} := \{M(T) : T \text{ a theory in } \mathcal{L}\}$, the set of definable model sets.

Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \mathbf{D}_{\mathcal{L}}$, $\mathbf{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.

An operation $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called definability preserving, (dp) or (μdp) in short, iff for all $X \in \mathbf{D}_{\mathcal{L}} \cap \mathcal{Y}$ $f(X) \in \mathbf{D}_{\mathcal{L}}$.

We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ - as needed for theory revision - with the obvious meaning.

\vdash will be classical derivability, and

$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under \vdash .

$Con(\cdot)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is classical consistent, likewise for $Con(T)$. $Con(T, T')$ will stand for $Con(T \cup T')$, etc.

Given a consequence relation \vdash , we define

$\overline{\overline{T}} := \{\phi : T \vdash \phi\}$.

(There is no fear of confusion with \overline{T} , as it just is not useful to close twice under classical logic.)

$T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.

If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m)$, $m \in M_{\mathcal{L}}$.

Definition 3.4

We introduce here formally a list of properties of set functions on the algebraic side, and their corresponding logical rules on the other side.

Recall that $\overline{T} := \{\phi : T \vdash \phi\}$, $\overline{\overline{T}} := \{\phi : T \vdash \phi\}$, where \vdash is classical consequence, and \vdash any other consequence.

We show, wherever adequate, in parallel the formula version in the left column, the theory version in the middle column, and the semantical or algebraic counterpart in the right column. The algebraic counterpart gives conditions for a function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, where U is some set, and $\mathcal{Y} \subseteq \mathcal{P}(U)$.

When the formula version is not commonly used, we omit it, as we normally work only with the theory version.

Intuitively, A and B in the right hand side column stand for $M(\phi)$ for some formula ϕ , whereas X, Y stand for $M(T)$ for some theory T .

Basics		
(AND) $\phi \sim \psi, \phi \sim \psi' \Rightarrow$ $\phi \sim \psi \wedge \psi'$	(AND) $T \sim \psi, T \sim \psi' \Rightarrow$ $T \sim \psi \wedge \psi'$	Closure under finite intersection
(OR) $\phi \sim \psi, \phi' \sim \psi \Rightarrow$ $\phi \vee \phi' \sim \psi$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ OR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(wOR) $\phi \sim \psi, \phi' \sim \psi \Rightarrow$ $\phi \vee \phi' \sim \psi$	(wOR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ wOR) $f(X \cup Y) \subseteq f(X) \cup Y$
(disjOR) $\phi \vdash \neg \phi', \phi \sim \psi,$ $\phi' \sim \psi \Rightarrow \phi \vee \phi' \sim \psi$	(disjOR) $\neg \text{Con}(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ disjOR) $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(LLE) Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \sim \psi \Rightarrow$ $\phi' \sim \psi$	(LLE) $\overline{\overline{T}} = \overline{\overline{T'}} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$	trivially true
(RW) Right Weakening $\phi \sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $\phi \sim \psi'$	(RW) $T \sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $T \sim \psi'$	upward closure
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ is classically closed	trivially true
(SC) Supraclassicality $\phi \vdash \psi \Rightarrow \phi \sim \psi$	(SC) $\overline{\overline{T}} \subseteq \overline{\overline{T}}$	($\mu \subseteq$) $f(X) \subseteq X$
(REF) Reflexivity $\Delta, \alpha \sim \alpha$		
(CP) Consistency Preservation $\phi \sim \perp \Rightarrow \phi \vdash \perp$	(CP) $T \sim \perp \Rightarrow T \vdash \perp$	($\mu \emptyset$) $f(X) = \emptyset \Rightarrow X = \emptyset$
		($\mu \emptyset \text{fin}$) $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\phi \cup \{\phi'\}}}$	$\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T \cup T'}}$	(μ PR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$
		(μ PR') $f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $\Delta \sim \alpha; \Delta, \alpha \sim \beta \Rightarrow$ $\Delta \sim \beta$	(CUT) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	(μ CUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$

Cumulativity		
(CM) Cautious Monotony $\phi \sim \psi, \phi \sim \psi' \Rightarrow$ $\phi \wedge \psi \sim \psi'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or $(ResM)$ Restricted Monotony $\Delta \sim \alpha, \beta \Rightarrow \Delta, \alpha \sim \beta$		$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow f(X \cap A) \subseteq B$
(CUM) Cumulativity $\phi \sim \psi \Rightarrow$ $(\phi \sim \psi' \Leftrightarrow \phi \wedge \psi \sim \psi')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	$(\mu \subseteq \supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$(\mu \subseteq \supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality		
$(RatM)$ Rational Monotony $\phi \sim \psi, \phi \not\sim \neg\psi' \Rightarrow$ $\phi \wedge \psi' \sim \psi$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'}} \cup T$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}} \cup T$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T}} \cup \overline{\overline{T'}} = \overline{\overline{T'}} \cup T$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
	$(Log \parallel)$ $\overline{\overline{T}} \vee \overline{\overline{T'}}$ is one of $\overline{\overline{T}},$ or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X), f(Y)$ or $f(X) \cup f(Y)$
	$(Log \cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T}} \vee \overline{\overline{T'}} \cup T')$	$(\mu \cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	$(Log \cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T}} \vee \overline{\overline{T'}} = \overline{\overline{T}}$	$(\mu \cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
		$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$

(PR) is also called infinite conditionalization - we choose the name for its central role for preferential structures (PR) or (μPR) .

The system of rules (AND) (OR) (LLE) (RW) (SC) (CP) (CM) (CUM) is also called system P (for preferential), adding $(RatM)$ gives the system R (for rationality or rankedness).

Roughly: Smooth preferential structures generate logics satisfying system P , ranked structures logics satisfying system R .

A logic satisfying (REF) , $(ResM)$, and (CUT) is called a consequence relation.

(LLE) and (CCL) will hold automatically, whenever we work with model sets.

(AND) is obviously closely related to filters, and corresponds to closure under finite intersections. (RW) corresponds to upward closure of filters.

More precisely, validity of both depend on the definition, and the direction we consider.

Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a principal filter: $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition: If $X = M(T)$, then $T \sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.

Conversely, we can define for $X = M(T)$

$\mathcal{X} := \{X' \subseteq X : \exists \phi (X' = X \cap M(\phi) \text{ and } T \vdash \phi)\}.$

(*AND*) then makes \mathcal{X} closed under finite intersections, (*RW*) makes \mathcal{X} upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete \mathcal{X} by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.

Alternatively, we can define

$\mathcal{X} := \{X' \subseteq X : \bigcap \{X \cap M(\phi) : T \vdash \phi\} \subseteq X'\}.$

(*SC*) corresponds to the choice of a subset.

(*CP*) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.

(*PR*) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ , T' for ψ , and $\phi \wedge \psi \vdash \sigma$, so $\phi \vdash \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.

(*CUM*) (whose most interesting half in our context is (*CM*)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold.

Definition 3.5

Fix $U \neq \emptyset$, and consider arbitrary X . Note that this X has not necessarily anything to do with U , or \mathcal{U} below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V - where V is the set theoretical universe we work in.

(A) Preferential models or structures.

(1) The version without copies:

A pair $\mathcal{M} := \langle U, \prec \rangle$ with U an arbitrary set, and \prec an arbitrary binary relation is called a preferential model or structure.

(2) The version with copies:

A pair $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ with \mathcal{U} an arbitrary set of pairs, and \prec an arbitrary binary relation is called a preferential model or structure.

If $\langle x, i \rangle \in \mathcal{U}$, then x is intended to be an element of U , and i the index of the copy.

We sometimes also need copies of the relation \prec , we will then replace \prec by one or several arrows α attacking non-minimal elements, e.g. $x \prec y$ will be written $\alpha : x \rightarrow y$, $\langle x, i \rangle \prec \langle y, i \rangle$ will be written $\alpha : \langle x, i \rangle \rightarrow \langle y, i \rangle$, and finally we might have $\langle \alpha, k \rangle : x \rightarrow y$ and $\langle \alpha, k \rangle : \langle x, i \rangle \rightarrow \langle y, i \rangle$, etc.

(B) Minimal elements, the functions $\mu_{\mathcal{M}}$

(1) The version without copies:

Let $\mathcal{M} := \langle U, \prec \rangle$, and define

$$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg \exists x' \in X \cap U. x' \prec x\}.$$

$\mu_{\mathcal{M}}(X)$ is called the set of minimal elements of X (in \mathcal{M}).

(2) The version with copies:

Let $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ be as above. Define

$$\mu_{\mathcal{M}}(X) := \{x \in X : \exists \langle x, i \rangle \in \mathcal{U}. \neg \exists \langle x', i' \rangle \in \mathcal{U} (x' \in X \wedge \langle x', i' \rangle \prec \langle x, i \rangle)\}.$$

Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of minimal elements of X in the structure. If the context is clear, we will also write just μ .

We sometimes say that $\langle x, i \rangle$ “kills” or “minimizes” $\langle y, j \rangle$ if $\langle x, i \rangle \prec \langle y, j \rangle$. By abuse of

language we also say a set X kills or minimizes a set Y if for all $\langle y, j \rangle \in \mathcal{U}$, $y \in Y$ there is $\langle x, i \rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x, i \rangle \prec \langle y, j \rangle$.

\mathcal{M} is also called injective or 1-copy, iff there is always at most one copy $\langle x, i \rangle$ for each x . Note that the existence of copies corresponds to a non-injective labelling function - as is often used in nonclassical logic, e.g. modal logic.

We say that \mathcal{M} is transitive, irreflexive, etc., iff \prec is.

Note that $\mu(X)$ might well be empty, even if X is not.

Definition 3.6

We define the consequence relation of a preferential structure for a given propositional language \mathcal{L} .

(A)

(1) If m is a classical model of a language \mathcal{L} , we say by abuse of language

$\langle m, i \rangle \models \phi$ iff $m \models \phi$,

and if X is a set of such pairs, that

$X \models \phi$ iff for all $\langle m, i \rangle \in X$ $m \models \phi$.

(2) If \mathcal{M} is a preferential structure, and X is a set of \mathcal{L} -models for a classical propositional language \mathcal{L} , or a set of pairs $\langle m, i \rangle$, where the m are such models, we call \mathcal{M} a classical preferential structure or model.

(B)

Validity in a preferential structure, or the semantical consequence relation defined by such a structure:

Let \mathcal{M} be as above.

We define:

$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e. $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

\mathcal{M} will be called definability preserving iff for all $X \in \mathbf{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \mathbf{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\mathbf{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 3.7

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, \mathcal{Y} will be $\mathbf{D}_{\mathcal{L}}$.)

A preferential structure \mathcal{M} is called \mathcal{Y} -smooth iff in every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X . More precisely:

(1) The version without copies:

If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X)$. $x' \prec x$.

(2) The version with copies:

If $x \in X \in \mathcal{Y}$, and $\langle x, i \rangle \in \mathcal{U}$, then either there is no $\langle x', i' \rangle \in \mathcal{U}$, $x' \in X$, $\langle x', i' \rangle \prec \langle x, i \rangle$ or there is $\langle x', i' \rangle \in \mathcal{U}$, $\langle x', i' \rangle \prec \langle x, i \rangle$, $x' \in X$, s.t. there is no $\langle x'', i'' \rangle \in \mathcal{U}$, $x'' \in X$, with $\langle x'', i'' \rangle \prec \langle x', i' \rangle$.

When considering the models of a language \mathcal{L} , \mathcal{M} will be called smooth iff it is $\mathbf{D}_{\mathcal{L}}$ -smooth; $\mathbf{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set \mathcal{Y} is, the stronger the condition \mathcal{Y} -smoothness will be.

Fact 3.1

Let \prec be an irreflexive, binary relation on X , then the following two conditions are equivalent:

- (1) There is Ω and an irreflexive, total, binary relation \prec' on Ω and a function $f : X \rightarrow \Omega$ s.t. $x \prec y \leftrightarrow f(x) \prec' f(y)$ for all $x, y \in X$.
- (2) Let $x, y, z \in X$ and $x \perp y$ wrt. \prec (i.e. neither $x \prec y$ nor $y \prec x$), then $z \prec x \rightarrow z \prec y$ and $x \prec z \rightarrow y \prec z$.

□

Definition 3.8

We call an irreflexive, binary relation \prec on X , which satisfies (1) (equivalently (2)) of Fact 3.1, ranked. By abuse of language, we also call a preferential structure $\langle X, \prec \rangle$ ranked, iff \prec is.

Definition 3.9

We have the usual framework of preferential structures, i.e. either a set with a possibly non-injective labelling function, or, equivalently, a set of possible worlds with copies. The relation of the preferential structure will be fixed, and will not depend on the point m from where we look at it.

Next, we have a set \mathbf{A} , and a finite, disjoint cover $A_i : i < n$ of \mathbf{A} , with a relation “of quality” $<$, \mathbf{A} will denote the A_i (and thus \mathbf{A}), and $<$, i.e. $\mathcal{A} = \langle \{A_i : i \in I\}, < \rangle$.

By Fact 3.24, we may assume that all A_i are described by a formula.

Finally, we have $\mathbf{B} \subseteq \mathbf{A}$, the subset of “good” elements of \mathbf{A} - which we also assume to be described by a formula.

In addition, we have a binary relation of accessibility, R , which we assume transitive - modal operators will be defined relative to R . R determines which part of the preferential structure is visible.

Let $R(s) := \{t : sRt\}$.

Definition 3.10

We repeat here from the introduction, and assume $A_i = M(\alpha_i)$, $B = M(\beta)$, and μ expresses the minimality of the preferential structure.

$$t \models \alpha_i > \beta \Leftrightarrow \mu(A_i) \cap R(t) \subseteq B,$$

we will also abuse notation and just write

$$t \models A_i > B \text{ in this case.}$$

We then define:

$$t \models \mathcal{C} \text{ iff at the smallest } i \text{ s.t. } \mu(A_i) \cap R(t) \neq \emptyset, \mu(A_i) \cap R(t) \subseteq \mathbf{B} \text{ holds.}$$

This motivates the following:

Definition 3.11

Let \mathbf{A} be a fixed set, and \mathcal{A} a finite, totally ordered (by $<$) disjoint cover by non-empty subsets of \mathbf{A} .

For $x \in \mathbf{A}$, let $rg(x)$ the unique $A \in \mathcal{A}$ such that $x \in A$, so $rg(x) < rg(y)$ is defined in the natural way.

A preferential structure $\langle \mathcal{X}, \prec \rangle$ (\mathcal{X} a set of pairs $\langle x, i \rangle$) is called \mathcal{A} -ranked iff for all $x, x' \in \mathcal{X}$ $rg(x) < rg(x')$ implies $\langle x, i \rangle \prec \langle x', i' \rangle$ for all $\langle x, i \rangle, \langle x', i' \rangle \in \mathcal{X}$.

Note that automatically for $X \subseteq \mathbf{A}$, $\mu(X) \subseteq A_j$ when j is the smallest i s.t. $X \cap A_i \neq \emptyset$.

The idea is now to make the A_i the layers, and “trigger” the first layer A_j s.t. $\mu(A_j) \cap R(x) \neq \emptyset$, and check whether $\mu(A_j) \cap R(x) \subseteq B_j$. A suitable ranked structure will automatically find this A_j .

More definitions and results for such \mathcal{A} and \mathcal{C} will be found in Section 4.

3.2 Discussion

The not necessarily smooth and the smooth case will be treated differently.

Strangely, the smooth case is simpler, as an added new layer in the proof settles it. Yet, this is not surprising when looking closer, as minimal elements never have higher rank, and we know from (μCUM) that minimizing by minimal elements suffices. All we have to add that any element in the minimal layer minimizes any element higher up.

In the simple, not necessarily smooth, case, we have to go deeper into the original proof to obtain the result.

The following idea, inspired by the treatment of the smooth case, will not work: Instead of minimizing by arbitrary elements, minimize only by elements of minimal rank, as the following example shows. If it worked, we might add just another layer to the original proof without $(\mu\mathcal{A})$, (see Definition 3.12), as in the smooth case.

Example 3.1

Consider the base set $\{a, b, c\}$, $\mu(\{a, b, c\}) = \{b\}$, $\mu(\{a, b\}) = \{a, b\}$, $\mu(\{a, c\}) = \emptyset$, $\mu(\{b, c\}) = \{b\}$, \mathcal{A} defined by $\{a, b\} < \{c\}$.

Obviously, $(\mu\mathcal{A})$ is satisfied. μ can be represented by the (not transitive!) relation $a \prec c \prec a$, $b \prec c$, which is \mathcal{A} -ranked.

But trying to minimize a in $\{a, b, c\}$ in the minimal layer will lead to $b \prec a$, and thus $a \notin \mu(\{a, b\})$, which is wrong.

□

Both proofs are self contained, the proof of the smooth case is stronger than the one published in [Sch04], as it does not need closure under finite intersections. The proofs of the general and transitive general case are adaptations of earlier proofs by the second author, but the basic ideas are not new, and were published before - see e.g. [Sch04], or [Sch92]. The proofs for the smooth case are slightly stronger than those published before (see again e.g. [Sch04]), as we work without closure under finite intersection. The rest is almost verbatim the same, and we only add a supplementary layer in the end (Fact 3.11), which will make the construction \mathcal{A} -ranked.

In the following, we will assume the partition \mathcal{A} to be given. We could also construct it from the properties of μ , but this would need stronger closure properties of the domain. The construction of \mathcal{A} is more difficult than the construction of the ranking in fully ranked structures, as $x \in \mu(X)$, $y \in X - \mu(X)$ will guarantee only $rg(x) \leq rg(y)$, and not $rg(x) < rg(y)$, as is the case in the latter situation. This corresponds to the separate treatment of the α and other formulas in the logical version, discussed in Section 3.5.

3.3 \mathcal{A} -ranked general and transitive structures

We will show here the following representation results:

Let \mathcal{A} be given.

An operation $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ is representable by an \mathcal{A} -ranked preferential structure iff μ satisfies $(\mu \subseteq)$, (μPR) , $(\mu \mathcal{A})$ (Proposition 3.5), and, moreover, the structure can be chosen transitive (Proposition 3.7).

Note that we carefully avoid any unnecessary assumptions about the domain $\mathcal{Y} \subseteq \mathcal{P}(Z)$ of the function μ .

Definition 3.12

We define a new condition:

Let \mathcal{A} be given as defined in Definition ??.

$(\mu \mathcal{A})$ If $X \in \mathcal{Y}$, $A, A' \in \mathcal{A}$, $A < A'$, $X \cap A \neq \emptyset$, $X \cap A' \neq \emptyset$ then $\mu(X) \cap A' = \emptyset$.

This new condition will be central for the modified representation.

3.3.1 The basic, not necessarily transitive, case

Definition 3.13

For $x \in Z$, let $\mathcal{Y}_x := \{Y \in \mathcal{Y} : x \in Y - \mu(Y)\}$, $\Pi_x := \Pi \mathcal{Y}_x$.

Note that $\emptyset \notin \mathcal{Y}_x$, $\Pi_x \neq \emptyset$, and that $\Pi_x = \{\emptyset\}$ iff $\mathcal{Y}_x = \emptyset$.

Claim 3.2

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ satisfy $(\mu \subseteq)$ and (μPR) , and let $U \in \mathcal{Y}$. Then $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Pi_x \cdot \text{ran}(f) \cap U = \emptyset$.

Proof:

Case 1: $\mathcal{Y}_x = \emptyset$, thus $\Pi_x = \{\emptyset\}$. “ \rightarrow ”: Take $f := \emptyset$. “ \leftarrow ”: $x \in U \in \mathcal{Y}$, $\mathcal{Y}_x = \emptyset \rightarrow x \in \mu(U)$ by definition of \mathcal{Y}_x .

Case 2: $\mathcal{Y}_x \neq \emptyset$. “ \rightarrow ”: Let $x \in \mu(U) \subseteq U$. It suffices to show $Y \in \mathcal{Y}_x \rightarrow Y - U \neq \emptyset$. But if $Y \subseteq U$ and $Y \in \mathcal{Y}_x$, then $x \in Y - \mu(Y)$, contradicting (μPR) . “ \leftarrow ”: If $x \in U - \mu(U)$, then $U \in \mathcal{Y}_x$, so $\forall f \in \Pi_x \cdot \text{ran}(f) \cap U \neq \emptyset$. \square

Construction 3.1

Let $\mathcal{X} := \{< x, f > : x \in Z \wedge f \in \Pi_x\}$, and $< x', f' > < x, f > :\leftrightarrow x' \in \text{ran}(f)$. Let $\mathcal{Z} := < \mathcal{X}, < > \}$.

Claim 3.3

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ satisfy $(\mu \subseteq)$ and (μPR) , and let $U \in \mathcal{Y}$. Then $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Pi_x \cdot \text{ran}(f) \cap U = \emptyset$.

Proof:

Case 1: $\mathcal{Y}_x = \emptyset$, thus $\Pi_x = \{\emptyset\}$. “ \rightarrow ”: Take $f := \emptyset$. “ \leftarrow ”: $x \in U \in \mathcal{Y}$, $\mathcal{Y}_x = \emptyset \rightarrow x \in \mu(U)$ by definition of \mathcal{Y}_x .

Case 2: $\mathcal{Y}_x \neq \emptyset$. “ \rightarrow ”: Let $x \in \mu(U) \subseteq U$. It suffices to show $Y \in \mathcal{Y}_x \rightarrow Y - U \neq \emptyset$. But if $Y \subseteq U$ and $Y \in \mathcal{Y}_x$, then $x \in Y - \mu(Y)$, contradicting (μPR) . “ \leftarrow ”: If $x \in U - \mu(U)$, then $U \in \mathcal{Y}_x$, so $\forall f \in \Pi_x.ran(f) \cap U \neq \emptyset$. \square

Corollary 3.4

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ satisfy $(\mu \subseteq)$, (μPR) , $(\mu \mathcal{A})$, and let $U \in \mathcal{Y}$.

If $x \in U$ and $\exists x' \in U.rg(x') < rg(x)$, then $\forall f \in \Pi_x.ran(f) \cap U \neq \emptyset$.

Proof:

By $(\mu \mathcal{A})$ $x \notin \mu(U)$, thus by Claim 3.3 $\forall f \in \Pi_x.ran(f) \cap U \neq \emptyset$. \square

Proposition 3.5

Let \mathcal{A} be given.

An operation $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ is representable by an \mathcal{A} -ranked preferential structure iff μ satisfies $(\mu \subseteq)$, (μPR) , $(\mu \mathcal{A})$.

Proof:

One direction is trivial. The central argument is: If $a \prec b$ in X , and $X \subseteq Y$, then $a \prec b$ in Y , too.

We turn to the other direction. The preferential structure is defined in Construction 3.2, Claim 3.6 shows representation.

Construction 3.2

Let $\mathcal{X} := \{ \langle x, f \rangle : x \in Z \wedge f \in \Pi_x \}$, and $\langle x', f' \rangle \prec \langle x, f \rangle :\Leftrightarrow x' \in ran(f)$ or $rg(x') < rg(x)$.

Note that, as \mathcal{A} is given, we also know $rg(x)$.

Let $\mathcal{Z} := \langle \mathcal{X}, \prec \rangle$.

Obviously, \mathcal{Z} is \mathcal{A} -ranked.

Claim 3.6

For $U \in \mathcal{Y}$, $\mu(U) = \mu_{\mathcal{Z}}(U)$.

Proof:

By Claim 3.3, it suffices to show that for all $U \in \mathcal{Y}$ $x \in \mu_{\mathcal{Z}}(U) \leftrightarrow x \in U$ and $\exists f \in \Pi_x.ran(f) \cap U = \emptyset$. So let $U \in \mathcal{Y}$.

“ \rightarrow ”: If $x \in \mu_Z(U)$, then there is $\langle x, f \rangle$ minimal in $\mathcal{X}[U]$ - where $\mathcal{X}[U] := \{\langle x, i \rangle \in \mathcal{X} : x \in U\}$, so $x \in U$, and there is no $\langle x', f' \rangle \succ \langle x, f \rangle$, $x' \in U$, so by $\Pi_{x'} \neq \emptyset$ there is no $x' \in \text{ran}(f)$, $x' \in U$, but then $\text{ran}(f) \cap U = \emptyset$.

“ \leftarrow ”: If $x \in U$, and there is $f \in \Pi_x$, $\text{ran}(f) \cap U = \emptyset$, then by Corollary 3.4, there is no $x' \in U$, $\text{rg}(x') < \text{rg}(x)$, so $\langle x, f \rangle$ is minimal in $\mathcal{X}[U]$.

□ (Claim 3.6 and Proposition 3.5)

3.3.2 The transitive case

Proposition 3.7

Let \mathcal{A} be given.

An operation $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ is representable by an \mathcal{A} -ranked transitive preferential structure iff μ satisfies $(\mu \subseteq)$, (μPR) , $(\mu \mathcal{A})$.

Construction 3.3

- (1) For $x \in Z$, let T_x be the set of trees t_x s.t.
 - (a) all nodes are elements of Z ,
 - (b) the root of t_x is x ,
 - (c) $\text{height}(t_x) \leq \omega$,
 - (d) if y is an element in t_x , then there is $f \in \Pi_y := \Pi\{Y \in \mathcal{Y} : y \in Y - \mu(Y)\}$ s.t. the set of children of y is $\text{ran}(f) \cup \{y' \in Z : \text{rg}(y') < \text{rg}(y)\}$.
- (2) For $x, y \in Z$, $t_x \in T_x$, $t_y \in T_y$, set $t_x \triangleright t_y$ iff y is a (direct) child of the root x in t_x , and t_y is the subtree of t_x beginning at y .
- (3) Let $\mathcal{Z} := \langle \langle x, t_x \rangle : x \in Z, t_x \in T_x \rangle$, $\langle x, t_x \rangle \succ \langle y, t_y \rangle$ iff $t_x \triangleright t_y$.

Fact 3.8

- (1) The construction ends at some y iff $\mathcal{Y}_y = \emptyset$ and there is no y' s.t. $\text{rg}(y') < \text{rg}(y)$, consequently $T_x = \{x\}$ iff $\mathcal{Y}_x = \emptyset$ and there are no x' with lesser rang. (We identify the tree of height 1 with its root.)
- (2) We define a special tree tc_x for all x : For all nodes y in tc_x , the successors are as follows: if $\mathcal{Y}_y \neq \emptyset$, then z is an successor iff $z = y$ or $\text{rg}(z) < \text{rg}(y)$; if $\mathcal{Y}_y = \emptyset$, then z is an successor iff $\text{rg}(z) < \text{rg}(y)$. (In the first case, we make $f \in \Pi_y$ always choose y itself.) tc_x is an element of T_x . Thus, with (1), $T_x \neq \emptyset$ for any x . Note: $tc_x = x$ iff $\mathcal{Y}_x = \emptyset$ and x has minimal rang.
- (3) If $f \in \Pi_x$, then the tree tf_x with root x and otherwise composed of the subtrees tc_y for $y \in \text{ran}(f) \cup \{y' : \text{rg}(y') < \text{rg}(y)\}$ is an element of T_x . (Level 0 of tf_x has x as element, the t'_y s begin at level 1.)
- (4) If y is an element in t_x and t_y the subtree of t_x starting at y , then $t_y \in T_y$.
- (5) $\langle x, t_x \rangle \succ \langle y, t_y \rangle$ implies $y \in \text{ran}(f) \cup \{x' : \text{rg}(x') < \text{rg}(x)\}$ for some $f \in \Pi_x$.

□

Claim 3.9 shows basic representation.

Claim 3.9

$$\forall U \in \mathcal{Y}. \mu(U) = \mu_{\mathcal{Z}}(U)$$

Proof:

By Claim 3.3, it suffices to show that for all $U \in \mathcal{Y}$ $x \in \mu_{\mathcal{Z}}(U) \leftrightarrow x \in U \wedge \exists f \in \Pi_x. \text{ran}(f) \cap U = \emptyset$.

Fix $U \in \mathcal{Y}$.

“ \rightarrow ”: $x \in \mu_{\mathcal{Z}}(U) \rightarrow \text{ex. } \langle x, t_x \rangle$ minimal in $\mathcal{Z}[U]$, thus $x \in U$ and there is no $\langle y, t_y \rangle \in \mathcal{Z}$, $\langle y, t_y \rangle \prec \langle x, t_x \rangle$, $y \in U$. Let f define the first part of the set of children of the root x in t_x . If $\text{ran}(f) \cap U \neq \emptyset$, if $y \in U$ is a child of x in t_x , and if t_y is the subtree of t_x starting at y , then $t_y \in T_y$ and $\langle y, t_y \rangle \prec \langle x, t_x \rangle$, contradicting minimality of $\langle x, t_x \rangle$ in $\mathcal{Z}[U]$. So $\text{ran}(f) \cap U = \emptyset$.

“ \leftarrow ”: Let $x \in U$, and $\exists f \in \Pi_x. \text{ran}(f) \cap U = \emptyset$. By Corollary 3.4, there is no $x' \in U$, $rg(x') < rg(x)$. If $\mathcal{Y}_x = \emptyset$, then the tree tc_x has no \triangleright -successors in U , and $\langle x, tc_x \rangle$ is \succ -minimal in $\mathcal{Z}[U]$. If $\mathcal{Y}_x \neq \emptyset$ and $f \in \Pi_x$ s.t. $\text{ran}(f) \cap U = \emptyset$, then $\langle x, tf_x \rangle$ is again \succ -minimal in $\mathcal{Z}[U]$.

□

We consider now the transitive closure of \mathcal{Z} . (Recall that \prec^* denotes the transitive closure of \prec .) Claim 3.10 shows that transitivity does not destroy what we have achieved. The trees tf_x play a crucial role in the demonstration.

Claim 3.10

Let $\mathcal{Z}' := \{ \langle x, t_x \rangle : x \in \mathcal{Z}, t_x \in T_x \}, \langle x, t_x \rangle \succ \langle y, t_y \rangle \text{ iff } t_x \triangleright^* t_y \}$. Then $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$.

Proof:

Suppose there is $U \in \mathcal{Y}$, $x \in U$, $x \in \mu_{\mathcal{Z}}(U)$, $x \notin \mu_{\mathcal{Z}'}(U)$. Then there must be an element $\langle x, t_x \rangle \in \mathcal{Z}$ with no $\langle x, t_x \rangle \succ \langle y, t_y \rangle$ for any $y \in U$. Let $f \in \Pi_x$ determine the first part of the set of children of x in t_x , then $\text{ran}(f) \cap U = \emptyset$, consider tf_x . All elements $w \neq x$ of tf_x are already in $\text{ran}(f)$, or $rg(w) < rg(x)$ holds. (Note that the elements chosen by rang in tf_x continue by themselves or by another element of even smaller rang, but the rang order is transitive.) But all w s.t. $rg(w) < rg(x)$ were already successors at level 1 of x in tf_x . By Corollary 3.4, there is no $w \in U$, $rg(w) < rg(x)$. Thus, no element $\neq x$ of tf_x is in U . Thus there is no $\langle z, t_z \rangle \prec^* \langle x, tf_x \rangle$ in \mathcal{Z} with $z \in U$, so $\langle x, tf_x \rangle$ is minimal in $\mathcal{Z}'[U]$, contradiction.

□ (Claim 3.10 and Proposition 3.7)

3.4 \mathcal{A} -ranked smooth structures

All smooth cases have a simple solution. We use one of our existing proofs for the not necessarily \mathcal{A} -ranked case, and add one little result:

Fact 3.11

Let $(\mu\mathcal{A})$ hold, and let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a smooth preferential structure representing μ , i.e. $\mu = \mu_{\mathcal{Z}}$.

Suppose that

$\langle x, i \rangle \prec \langle y, j \rangle$ implies $rg(x) \leq rg(y)$.

Define $\mathcal{Z}' := \langle \mathcal{X}, \sqsubset \rangle$ where $\langle x, i \rangle \sqsubset \langle y, j \rangle$ iff $\langle x, i \rangle \prec \langle y, j \rangle$ or $rg(x) < rg(y)$.

Then \mathcal{Z}' is \mathcal{A} -ranked.

\mathcal{Z}' is smooth, too, and $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'} =: \mu'$.

In addition, if \prec is free from cycles, so is \sqsubset , if \prec is transitive, so is \sqsubset .

Proof:

\mathcal{A} -rankedness is trivial.

Suppose $\langle x, i \rangle$ is \prec -minimal, but not \sqsubset -minimal. Then there must be $\langle y, j \rangle \sqsubset \langle x, i \rangle$, $\langle y, j \rangle \not\prec \langle x, i \rangle$, $y \in X$, so $rg(y) < rg(x)$. By $(\mu\mathcal{A})$, all $x \in \mu(X)$ have minimal \mathcal{A} -rang among the elements of X , so this is impossible. Thus, μ -minimal elements stay μ' -minimal, so smoothness will also be preserved - remember that we increased the relation.

By prerequisite, there cannot be any cycle involving only \prec , but the rang order is free from cycles, too, and \prec respects the rang order, so \sqsubset is free from cycles.

Let \prec be transitive, so is the rang order. But if $\langle x, i \rangle \prec \langle y, j \rangle$ and $rg(y) < rg(z)$ for some $\langle z, k \rangle$, then by prerequisite $rg(x) \leq rg(y)$, so $rg(x) < rg(z)$, so $\langle x, i \rangle \sqsubset \langle z, k \rangle$ by definition. Likewise for $rg(x) < rg(y)$ and $\langle y, j \rangle \prec \langle z, k \rangle$.

□

All that remains to show then is that our constructions of smooth and of smooth and transitive structures satisfy the condition

$\langle x, i \rangle \prec \langle y, j \rangle$ implies $rg(x) \leq rg(y)$.

3.4.1 The basic smooth, not necessarily transitive case

We will show here the following representation result:

Proposition 3.12

Let \mathcal{A} be given.

Let \mathcal{Y} be closed under finite unions, and $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$. Then there is a \mathcal{Y} -smooth \mathcal{A} -ranked preferential structure \mathcal{Z} , s.t. for all $X \in \mathcal{Y}$ $\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq)$, (μPR) , (μCUM) , $(\mu\mathcal{A})$.

To prove Proposition 3.12, we first show and prove:

Proposition 3.13

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(U)$ satisfy $(\mu \subseteq)$, (μPR) , and (μCUM) , and the domain $\mathcal{Y} (\cup)$.

Then there is a \mathcal{Y} -smooth preferential structure \mathcal{X} s.t. $\mu = \mu_{\mathcal{X}}$.

Proof

Outline: We first define a structure \mathcal{Z} (in a way very similar to Construction 3.1) which represents μ , but is not necessarily \mathcal{Y} -smooth, refine it to \mathcal{Z}' and show that \mathcal{Z}' represents μ too, and that \mathcal{Z}' is \mathcal{Y} -smooth.

In the structure \mathcal{Z}' , all pairs destroying smoothness in \mathcal{Z} are successively repaired, by adding minimal elements: If $\langle y, j \rangle$ is not minimal, and has no minimal $\langle x, i \rangle$ below it, we just add one such $\langle x, i \rangle$. As the repair process might itself generate such “bad” pairs, the process may have to be repeated infinitely often. Of course, one has to take care that the representation property is preserved.

The proof given is close to the minimum one has to show (except that we avoid $H(U)$, instead of U - as was done in the old proof of [Sch96-1]). We could simplify further, we do not, in order to stay closer to the construction that is really needed. The reader will find the simplification as building block of the proof of the transitive case. (In the simplified proof, we would consider for x, U s.t. $x \in \mu(U)$ the pairs $\langle x, g_U \rangle$ with $g_U \in \Pi\{\mu(U \cup Y) : x \in Y \not\subseteq H(U)\}$, giving minimal elements. For the U s.t. $x \in U - \mu(U)$, we would choose $\langle x, g \rangle$ s.t. $g \in \Pi\{\mu(Y) : x \in Y \in \mathcal{Y}\}$ with $\langle x', g'_U \rangle \prec \langle x, g \rangle$ for $\langle x', g'_U \rangle$ as above.)

Construction 3.4 represents μ . The structure will not yet be smooth, we will mend it afterwards in Construction 3.5.

The constructions

\mathcal{Y} will be closed under finite unions throughout this Section. We first define $H(U)$, and show some facts about it. $H(U)$ has an important role, for the following reason: If $u \in \mu(U)$, but $u \in X - \mu(X)$, then there is $x \in \mu(X) - H(U)$. Consequently, to kill minimality of u in X , we can choose $x \in \mu(X) - H(U)$, $x \prec u$, without interfering with u 's minimality in U . Moreover, if $x \in Y - \mu(Y)$, then, by $x \notin H(U)$, $\mu(Y) \not\subseteq H(U)$, so we can kill minimality of x in Y by choosing some $y \notin H(U)$. Thus, even in the transitive case, we can leave U to destroy minimality of u in some X , without ever having to come back into U , it suffices to choose sufficiently far from U , i.e. outside $H(U)$. $H(U)$ is the right notion of “neighborhood”.

Note: Not all $z \in Z$ have to occur in our structure, therefore it is quite possible that $X \in \mathcal{Y}$, $X \neq \emptyset$, but $\mu_{\mathcal{Z}}(X) = \emptyset$. This is why we have introduced the set K in Definition 3.15 and such X will be subsets of $Z-K$.

Let now $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$.

Definition 3.14

Define $H(U) := \bigcup \{X : \mu(X) \subseteq U\}$.

Definition 3.15

Let $K := \{x \in Z : \exists X \in \mathcal{Y}. x \in \mu(X)\}$

Fact 3.14

$(\mu \subseteq) + (\mu PR) + (\mu CUM) + (\cup)$ entail:

- (1) $\mu(A) \subseteq B \rightarrow \mu(A \cup B) = \mu(B)$
- (2) $\mu(X) \subseteq U, U \subseteq Y \rightarrow \mu(Y \cup X) = \mu(Y)$
- (3) $\mu(X) \subseteq U, U \subseteq Y \rightarrow \mu(Y) \cap X \subseteq \mu(U)$
- (4) $\mu(X) \subseteq U \rightarrow \mu(U) \cap X \subseteq \mu(X)$
- (5) $U \subseteq A, \mu(A) \subseteq H(U) \rightarrow \mu(A) \subseteq U$
- (6) Let $x \in K, Y \in \mathcal{Y}, x \in Y - \mu(Y)$, then $\mu(Y) \neq \emptyset$.

Proof:

- (1) $\mu(A) \subseteq B \rightarrow \mu(A \cup B) \subseteq \mu(A) \cup \mu(B) \subseteq B \rightarrow_{(\mu CUM)} \mu(B) = \mu(A \cup B)$.
- (2) trivial by (1).
- (3) $\mu(Y) \cap X = (\text{by (2)}) \mu(Y \cup X) \cap X \subseteq \mu(Y \cup X) \cap (X \cup U) \subseteq (\text{by } (\mu PR)) \mu(X \cup U) = (\text{by (1)}) \mu(U)$.
- (4) $\mu(U) \cap X = \mu(X \cup U) \cap X$ by (1) $\subseteq \mu(X)$ by (μPR)
- (5) Let $U \subseteq A, \mu(A) \subseteq H(U)$. So $\mu(A) = \bigcup \{\mu(A) \cap Y : \mu(Y) \subseteq U\} \subseteq \mu(U) \subseteq U$ by (3).
- (6) Suppose $x \in \mu(X), \mu(Y) = \emptyset \rightarrow \mu(Y) \subseteq X$, so by (4) $Y \cap \mu(X) \subseteq \mu(Y)$, so $x \in \mu(Y)$.

□

The following Fact 3.15 contains the basic properties of μ and $H(U)$ which we will need for the representation construction.

Fact 3.15

Let A, U, U', Y and all A_i be in \mathcal{Y} . Let $(\mu \subseteq) + (\mu PR) + (\cup)$ hold.

- (1) $A = \bigcup \{A_i : i \in I\} \rightarrow \mu(A) \subseteq \bigcup \{\mu(A_i) : i \in I\}$,
- (2) $U \subseteq H(U)$, and $U \subseteq U' \rightarrow H(U) \subseteq H(U')$,
- (3) $\mu(U \cup Y) - H(U) \subseteq \mu(Y)$.

If, in addition, (μCUM) holds, then we also have:

- (4) $U \subseteq A, \mu(A) \subseteq H(U) \rightarrow \mu(A) \subseteq U,$
- (5) $\mu(Y) \subseteq H(U) \rightarrow Y \subseteq H(U) \text{ and } \mu(U \cup Y) = \mu(U),$
- (6) $x \in \mu(U), x \in Y - \mu(Y) \rightarrow Y \not\subseteq H(U),$
- (7) $Y \not\subseteq H(U) \rightarrow \mu(U \cup Y) \not\subseteq H(U).$

Proof:

- (1) $\mu(A) \cap A_j \subseteq \mu(A_j) \subseteq \bigcup \mu(A_i),$ so by $\mu(A) \subseteq A = \bigcup A_i$ $\mu(A) \subseteq \bigcup \mu(A_i).$
- (2) trivial.
- (3) $\mu(U \cup Y) - H(U) \subseteq_{(2)} \mu(U \cup Y) - U \subseteq_{(\mu \subseteq)} \mu(U \cup Y) \cap Y \subseteq_{(\mu PR)} \mu(Y).$
- (4) This is Fact 3.14 (5).
- (5) Let $\mu(Y) \subseteq H(U),$ then by $\mu(U) \subseteq H(U)$ and (1) $\mu(U \cup Y) \subseteq \mu(U) \cup \mu(Y) \subseteq H(U),$ so by (4) $\mu(U \cup Y) \subseteq U$ and $U \cup Y \subseteq H(U).$ Moreover, $\mu(U \cup Y) \subseteq U \subseteq U \cup Y \rightarrow_{(\mu CUM)} \mu(U \cup Y) = \mu(U).$
- (6) If not, $Y \subseteq H(U),$ so $\mu(Y) \subseteq H(U),$ so $\mu(U \cup Y) = \mu(U)$ by (5), but $x \in Y - \mu(Y) \rightarrow_{(\mu PR)} x \notin \mu(U \cup Y) = \mu(U),$ *contradiction.*
- (7) $\mu(U \cup Y) \subseteq H(U) \rightarrow_{(5)} U \cup Y \subseteq H(U). \square$

Definition 3.16

For $x \in Z,$ let $\mathcal{W}_x := \{\mu(Y): Y \in \mathcal{Y} \wedge x \in Y - \mu(Y)\}, \Gamma_x := \Pi \mathcal{W}_x,$ and $K := \{x \in Z: \exists X \in \mathcal{Y}. x \in \mu(X)\}.$

Note that we consider here now $\mu(Y)$ in $\mathcal{W}_x,$ and not Y as in \mathcal{Y}_x in Definition 3.13.

Remark 3.16

Assume now $(\mu \subseteq), (\mu PR), (\mu CUM), (\cup)$ to hold.

- (1) $x \in K \rightarrow \Gamma_x \neq \emptyset,$
- (2) $g \in \Gamma_x \rightarrow \text{ran}(g) \subseteq K.$

Proof:

- (1) We have to show that $Y \in \mathcal{Y}, x \in Y - \mu(Y) \rightarrow \mu(Y) \neq \emptyset.$ This was shown in Fact 3.14 (6).

(2) By definition, $\mu(Y) \subseteq K$ for all $Y \in \mathcal{Y}$. \square

The following claim is the analogue of Claim 3.2 above.

Claim 3.17

Assume now $(\mu \subseteq)$, (μPR) , (μCUM) , (\cup) to hold.

Let $U \in \mathcal{Y}$, $x \in K$. Then

- (1) $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x.ran(f) \cap U = \emptyset$,
- (2) $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x.ran(f) \cap H(U) = \emptyset$.

Proof:

(1) Case 1: $\mathcal{W}_x = \emptyset$, thus $\Gamma_x = \{\emptyset\}$. “ \rightarrow ”: Take $f := \emptyset$. “ \leftarrow ”: $x \in U \in \mathcal{Y}$, $\mathcal{W}_x = \emptyset \rightarrow x \in \mu(U)$ by definition of \mathcal{W}_x .

Case 2: $\mathcal{W}_x \neq \emptyset$. “ \rightarrow ”: Let $x \in \mu(U) \subseteq U$. It suffices to show $Y \in \mathcal{W}_x \rightarrow \mu(Y) - H(U) \neq \emptyset$. But $Y \in \mathcal{W}_x \rightarrow x \in Y - \mu(Y) \rightarrow$ (by Fact 3.15, (6)) $Y \not\subseteq H(U) \rightarrow$ (by Fact 3.15, (5)) $\mu(Y) \not\subseteq H(U)$. “ \leftarrow ”: If $x \in U - \mu(U)$, $U \in \mathcal{W}_x$, moreover $\Gamma_x \neq \emptyset$ by Remark 3.16, (1) and thus (or by the same argument) $\mu(U) \neq \emptyset$, so $\forall f \in \Gamma_x.ran(f) \cap U \neq \emptyset$.

(2): The proof is verbatim the same as for (1). \square

Proof: (Prop. 6.14)

Construction 3.4

(Construction of \mathcal{Z}) Let $\mathcal{X} := \{ \langle x, g \rangle : x \in K, g \in \Gamma_x \}$, $\langle x', g' \rangle \prec \langle x, g \rangle :\leftrightarrow x' \in ran(g)$,
 $\mathcal{Z} := \langle \mathcal{X}, \prec \rangle$.

Claim 3.18

$\forall U \in \mathcal{Y}. \mu(U) = \mu_{\mathcal{Z}}(U)$

Proof:

Case 1: $x \notin K$. Then $x \notin \mu(U)$ and $x \notin \mu_Z(U)$.

Case 2: $x \in K$. By Claim 3.17, (1) it suffices to show that for all $U \in \mathcal{Y}$ $x \in \mu_Z(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x. \text{ran}(f) \cap U = \emptyset$. Fix $U \in \mathcal{Y}$. “ \rightarrow ”: $x \in \mu_Z(U) \rightarrow$ ex. $\langle x, f \rangle$ minimal in $\mathcal{X}[U]$, thus $x \in U$ and there is no $\langle x', f' \rangle \prec \langle x, f \rangle$, $x' \in U$, $x' \in K$. But if $x' \in K$, then by Remark 3.16, (1), $\Gamma_{x'} \neq \emptyset$, so we find suitable f' . Thus, $\forall x' \in \text{ran}(f). x' \notin U$ or $x' \notin K$. But $\text{ran}(f) \subseteq K$, so $\text{ran}(f) \cap U = \emptyset$. “ \leftarrow ”: If $x \in U$, $f \in \Gamma_x$ s.t. $\text{ran}(f) \cap U = \emptyset$, then $\langle x, f \rangle$ is minimal in $\mathcal{X}[U]$.

□

We now construct the refined structure \mathcal{Z}' .

Construction 3.5

(Construction of \mathcal{Z}')

σ is called x -admissible sequence iff

1. σ is a sequence of length $\leq \omega$, $\sigma = \{\sigma_i : i \in \omega\}$,
2. $\sigma_0 \in \Pi\{\mu(Y) : Y \in \mathcal{Y} \wedge x \in Y - \mu(Y)\}$,
3. $\sigma_{i+1} \in \Pi\{\mu(X) : X \in \mathcal{Y} \wedge x \in \mu(X) \wedge \text{ran}(\sigma_i) \cap X \neq \emptyset\}$.

By 2., σ_0 minimizes x , and by 3., if $x \in \mu(X)$, and $\text{ran}(\sigma_i) \cap X \neq \emptyset$, i.e. we have destroyed minimality of x in X , x will be above some y minimal in X to preserve smoothness.

Let Σ_x be the set of x -admissible sequences, for $\sigma \in \Sigma_x$ let $\widehat{\sigma} := \bigcup\{\text{ran}(\sigma_i) : i \in \omega\}$. Note that by the argument in the proof of Remark 3.16, (1), $\Sigma_x \neq \emptyset$, if $x \in K$.

Let $\mathcal{X}' := \{\langle x, \sigma \rangle : x \in K \wedge \sigma \in \Sigma_x\}$ and $\langle x', \sigma' \rangle \succ' \langle x, \sigma \rangle :\leftrightarrow x' \in \widehat{\sigma}$. Finally, let $\mathcal{Z}' := \langle \mathcal{X}', \prec' \rangle$, and $\mu' := \mu_{\mathcal{Z}'}$.

It is now easy to show that \mathcal{Z}' represents μ , and that \mathcal{Z}' is smooth. For $x \in \mu(U)$, we construct a special x -admissible sequence $\sigma^{x,U}$ using the properties of $H(U)$.

Claim 3.19

For all $U \in \mathcal{Y}$ $\mu(U) = \mu_Z(U) = \mu'(U)$.

Proof:

If $x \notin K$, then $x \notin \mu_Z(U)$, and $x \notin \mu'(U)$ for any U . So assume $x \in K$. If $x \in U$ and $x \notin \mu_Z(U)$, then for all $\langle x, f \rangle \in \mathcal{X}$, there is $\langle x', f' \rangle \in \mathcal{X}$ with $\langle x', f' \rangle \prec \langle x, f \rangle$ and $x' \in U$. Let now $\langle x, \sigma \rangle \in \mathcal{X}'$,

then $\langle x, \sigma_0 \rangle \in \mathcal{X}$, and let $\langle x', f' \rangle \prec \langle x, \sigma_0 \rangle$ in \mathcal{Z} with $x' \in U$. As $x' \in K$, $\Sigma_{x'} \neq \emptyset$, let $\sigma' \in \Sigma_{x'}$. Then $\langle x', \sigma' \rangle \prec \langle x, \sigma \rangle$ in \mathcal{Z}' . Thus $x \notin \mu'(U)$. Thus, for all $U \in \mathcal{Y}$, $\mu'(U) \subseteq \mu_{\mathcal{Z}}(U) = \mu(U)$.

It remains to show $x \in \mu(U) \rightarrow x \in \mu'(U)$.

Assume $x \in \mu(U)$ (so $x \in K$), $U \in \mathcal{Y}$, we will construct minimal σ , i.e. show that there is $\sigma^{x,U} \in \Sigma_x$ s.t. $\widehat{\sigma^{x,U}} \cap U = \emptyset$. We construct this $\sigma^{x,U}$ inductively, with the stronger property that $\text{ran}(\sigma_i^{x,U}) \cap H(U) = \emptyset$ for all $i \in \omega$.

$\sigma_0^{x,U}$: $x \in \mu(U)$, $x \in Y - \mu(Y) \rightarrow \mu(Y) - H(U) \neq \emptyset$ by Fact 3.15, (6) + (5). Let $\sigma_0^{x,U} \in \Pi\{\mu(Y) - H(U) : Y \in \mathcal{Y}, x \in Y - \mu(Y)\}$, so $\text{ran}(\sigma_0^{x,U}) \cap H(U) = \emptyset$.

$\sigma_i^{x,U} \rightarrow \sigma_{i+1}^{x,U}$: By induction hypothesis, $\text{ran}(\sigma_i^{x,U}) \cap H(U) = \emptyset$. Let $X \in \mathcal{Y}$ be s.t. $x \in \mu(X)$, $\text{ran}(\sigma_i^{x,U}) \cap X \neq \emptyset$. Thus $X \not\subseteq H(U)$, so $\mu(U \cup X) - H(U) \neq \emptyset$ by Fact 3.15, (7). Let $\sigma_{i+1}^{x,U} \in \Pi\{\mu(U \cup X) - H(U) : X \in \mathcal{Y}, x \in \mu(X), \text{ran}(\sigma_i^{x,U}) \cap X \neq \emptyset\}$, so $\text{ran}(\sigma_{i+1}^{x,U}) \cap H(U) = \emptyset$. As $\mu(U \cup X) - H(U) \subseteq \mu(X)$ by Fact 3.15, (3), the construction satisfies the x -admissibility condition. \square

We now show:

Claim 3.20

\mathcal{Z}' is \mathcal{Y} -smooth.

Proof:

Let $X \in \mathcal{Y}$, $\langle x, \sigma \rangle \in \mathcal{X}' \upharpoonright X$.

Case 1, $x \in X - \mu(X)$: Then $\text{ran}(\sigma_0) \cap \mu(X) \neq \emptyset$, let $x' \in \text{ran}(\sigma_0) \cap \mu(X)$. Moreover, $\mu(X) \subseteq K$. Then for all $\langle x', \sigma' \rangle \in \mathcal{X}'$ $\langle x', \sigma' \rangle \prec \langle x, \sigma \rangle$. But $\langle x', \sigma^{x',X} \rangle$ as constructed in the proof of Claim 3.19 is minimal in $\mathcal{X}' \upharpoonright X$.

Case 2, $x \in \mu(X) = \mu_{\mathcal{Z}}(X) = \mu'(X)$: If $\langle x, \sigma \rangle$ is minimal in $\mathcal{X}' \upharpoonright X$, we are done. So suppose there is $\langle x', \sigma' \rangle \prec \langle x, \sigma \rangle$, $x' \in X$. Thus $x' \in \widehat{\sigma}$. Let $x' \in \text{ran}(\sigma_i)$. So $x \in \mu(X)$ and $\text{ran}(\sigma_i) \cap X \neq \emptyset$. But $\sigma_{i+1} \in \Pi\{\mu(X') : X' \in \mathcal{Y} \wedge x \in \mu(X') \wedge \text{ran}(\sigma_i) \cap X' \neq \emptyset\}$, so X is one of the X' , moreover $\mu(X) \subseteq K$, so there is $x'' \in \mu(X) \cap \text{ran}(\sigma_{i+1}) \cap K$, so for all $\langle x'', \sigma'' \rangle \in \mathcal{X}'$ $\langle x'', \sigma'' \rangle \prec \langle x, \sigma \rangle$. But again $\langle x'', \sigma^{x'',X} \rangle$ as constructed in the proof of Claim 3.19 is minimal in $\mathcal{X}' \upharpoonright X$. \square

Proof of Proposition 3.12

Consider the construction in the proof of Proposition 3.13. We have to show that it respects the rang order with respect to \mathcal{A} , i.e. that $\langle x', \sigma' \rangle \prec' \langle x, \sigma \rangle$ implies $rg(x') \leq rg(x)$. This is easy: By definition, $x' \in \bigcup \{ran(\sigma_i) : i \in \omega\}$. If $x' \in ran(\sigma_0)$, then for some Y $x' \in \mu(Y)$, $x \in Y - \mu(Y)$, so $rg(x') \leq rg(x)$ by $(\mu\mathcal{A})$. If $x' \in ran(\sigma_i)$, $i > 0$, then for some X $x', x \in \mu(X)$, so $rg(x) = rg(x')$ by $(\mu\mathcal{A})$.

□ (Proposition 3.12)

3.4.2 The transitive smooth case

We will show here the transitive analogon of Proposition 3.12:

Proposition 3.21

Let \mathcal{A} be given.

Let \mathcal{Y} be closed under finite unions, and $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$. Then there is a \mathcal{Y} –smooth \mathcal{A} –ranked transitive preferential structure \mathcal{Z} , s.t. for all $X \in \mathcal{Y}$ $\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq)$, (μPR) , (μCUM) , $(\mu\mathcal{A})$.

To prove Proposition 3.21, we first show and prove:

Proposition 3.22

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(U)$ satisfy $(\mu \subseteq)$, (μPR) , and (μCUM) , and the domain $\mathcal{Y} (\cup)$.

Then there is a transitive \mathcal{Y} –smooth preferential structure \mathcal{X} s.t. $\mu = \mu_{\mathcal{X}}$.

Proof

Discussion Smooth-Trans

In a certain way, it is not surprising that transitivity does not impose stronger conditions in the smooth case either. Smoothness is itself a weak kind of transitivity: If an element is not minimal, then there is a minimal element below it, i.e., $x \succ y$ with y not minimal is possible, there is $z' \prec y$, but then there is z minimal with $x \succ z$. This is “almost” $x \succ z'$, transitivity.

To obtain representation, we will combine here the ideas of the smooth, but not necessarily transitive case with those of the general transitive case - as the reader will have suspected. Thus, we will index again with trees, and work with (suitably adapted) admissible sequences for the construction of the trees. In the construction of the admissible sequences, we were careful to repair all damage done in previous

steps. We have to add now reparation of all damage done by using transitivity, i.e., the transitivity of the relation might destroy minimality, and we have to construct minimal elements below all elements for which we thus destroyed minimality. Both cases are combined by considering immediately all Y s.t. $x \in Y - H(U)$. Of course, the properties described in Fact 3.15 play again a central role.

The (somewhat complicated) construction will be commented on in more detail below.

Note that even beyond Fact 3.15, closure of the domain under finite unions is used in the construction of the trees. This - or something like it - is necessary, as we have to respect the hulls of all elements treated so far (the predecessors), and not only of the first element, because of transitivity. For the same reason, we need more bookkeeping, to annotate all the hulls (or the union of the respective U 's) of all predecessors to be respected.

To summarize: we combine the ideas from the transitive general case and the simple smooth case, using the crucial Fact 3.15 to show that the construction goes through. The construction leaves still some freedom, and modifications are possible as indicated below in the course of the proof.

Recall that \mathcal{Y} will be closed under finite unions in this section, and let again $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$.

We have to adapt Construction 3.5 (x-admissible sequences) to the transitive situation, and to our construction with trees. If $\langle \emptyset, x \rangle$ is the root, $\sigma_0 \in \Pi\{\mu(Y) : x \in Y - \mu(Y)\}$ determines some children of the root. To preserve smoothness, we have to compensate and add other children by the $\sigma_{i+1} : \sigma_{i+1} \in \Pi\{\mu(X) : x \in \mu(X), \text{ran}(\sigma_i) \cap X \neq \emptyset\}$. On the other hand, we have to pursue the same construction for the children so constructed. Moreover, these indirect children have to be added to those children of the root, which have to be compensated (as the first children are compensated by σ_1) to preserve smoothness. Thus, we build the tree in a simultaneous vertical and horizontal induction.

This construction can be simplified, by considering immediately all $Y \in \mathcal{Y}$ s.t. $x \in Y \not\subseteq H(U)$ - independent of whether $x \notin \mu(Y)$ (as done in σ_0), or whether $x \in \mu(Y)$, and some child y constructed before is in Y (as done in the σ_{i+1}), or whether $x \in \mu(Y)$, and some indirect child y of x is in Y (to take care of transitivity, as indicated above). We make this simplified construction.

There are two ways to proceed. First, we can take as \triangleleft^* in the trees the transitive closure of \triangleleft . Second, we can deviate from the idea that children are chosen by selection functions f , and take nonempty subsets of elements instead, making more elements children than in the first case. We take the first alternative, as it is more in the spirit of the construction.

We will suppose for simplicity that $Z = K$ - the general case is easy to obtain, but complicates the picture.

For each $x \in Z$, we construct trees t_x , which will be used to index different copies of x , and control the relation \prec .

These trees t_x will have the following form:

- (a) the root of t is $\langle \emptyset, x \rangle$ or $\langle U, x \rangle$ with $U \in \mathcal{Y}$ and $x \in \mu(U)$,
- (b) all other nodes are pairs $\langle Y, y \rangle$, $Y \in \mathcal{Y}$, $y \in \mu(Y)$,
- (c) $ht(t) \leq \omega$,
- (d) if $\langle Y, y \rangle$ is an element in t_x , then there is some $\mathcal{Y}(y) \subseteq \{W \in \mathcal{Y} : y \in W\}$, and $f \in \Pi\{\mu(W) : W \in \mathcal{Y}(y)\}$ s.t. the set of children of $\langle Y, y \rangle$ is $\{\langle Y \cup W, f(W) \rangle : W \in \mathcal{Y}(y)\}$.

The first coordinate is used for bookkeeping when constructing children, in particular for condition (d).

The relation \prec will essentially be determined by the subtree relation.

We first construct the trees t_x for those sets U where $x \in \mu(U)$, and then take care of the others. In the construction for the minimal elements, at each level $n > 0$, we may have several ways to choose a

selection function f_n , and each such choice leads to the construction of a different tree - we construct all these trees. (We could also construct only one tree, but then the choice would have to be made coherently for different x, U . It is simpler to construct more trees than necessary.)

We control the relation by indexing with trees, just as it was done in the not necessarily smooth case before.

Definition 3.17

If t is a tree with root $\langle a, b \rangle$, then t/c will be the same tree, only with the root $\langle c, b \rangle$.

Construction 3.6

(A) The set T_x of trees t for fixed x :

(1) Construction of the set $T\mu_x$ of trees for those sets $U \in \mathcal{Y}$, where $x \in \mu(U)$:

Let $U \in \mathcal{Y}$, $x \in \mu(U)$. The trees $t_{U,x} \in T\mu_x$ are constructed inductively, observing simultaneously:

If $\langle U_{n+1}, x_{n+1} \rangle$ is a child of $\langle U_n, x_n \rangle$, then (a) $x_{n+1} \in \mu(U_{n+1}) - H(U_n)$, and (b) $U_n \subseteq U_{n+1}$.

Set $U_0 := U$, $x_0 := x$.

Level 0: $\langle U_0, x_0 \rangle$.

Level $n \rightarrow n+1$: Let $\langle U_n, x_n \rangle$ be in level n . Suppose $Y_{n+1} \in \mathcal{Y}$, $x_n \in Y_{n+1}$, and $Y_{n+1} \not\subseteq H(U_n)$. Note that $\mu(U_n \cup Y_{n+1}) - H(U_n) \neq \emptyset$ by Fact 3.15, (7), and $\mu(U_n \cup Y_{n+1}) - H(U_n) \subseteq \mu(Y_{n+1})$ by Fact 3.15, (3). Choose $f_{n+1} \in \Pi\{\mu(U_n \cup Y_{n+1}) - H(U_n) : Y_{n+1} \in \mathcal{Y}, x_n \in Y_{n+1} \not\subseteq H(U_n)\}$ (for the construction of this tree, at this element), and let the set of children of $\langle U_n, x_n \rangle$ be $\{\langle U_n \cup Y_{n+1}, f_{n+1}(Y_{n+1}) \rangle : Y_{n+1} \in \mathcal{Y}, x_n \in Y_{n+1} \not\subseteq H(U_n)\}$. (If there is no such Y_{n+1} , $\langle U_n, x_n \rangle$ has no children.) Obviously, (a) and (b) hold.

We call such trees U, x -trees.

(2) Construction of the set T'_x of trees for the nonminimal elements. Let $x \in Z$. Construct the tree t_x as follows (here, one tree per x suffices for all U):

Level 0: $\langle \emptyset, x \rangle$

Level 1: Choose arbitrary $f \in \Pi\{\mu(U) : x \in U \in \mathcal{Y}\}$. Note that $U \neq \emptyset \rightarrow \mu(U) \neq \emptyset$ by $Z = K$ (by Remark 3.16, (1)). Let $\{\langle U, f(U) \rangle : x \in U \in \mathcal{Y}\}$ be the set of children of $\langle \emptyset, x \rangle$. This assures that the element will be nonminimal.

Level > 1 : Let $\langle U, f(U) \rangle$ be an element of level 1, as $f(U) \in \mu(U)$, there is a $t_{U,f(U)} \in T\mu_{f(U)}$. Graft one of these trees $t_{U,f(U)} \in T\mu_{f(U)}$ at $\langle U, f(U) \rangle$ on the level 1. This assures that a minimal element will be below it to guarantee smoothness.

Finally, let $T_x := T\mu_x \cup T'_x$.

(B) The relation \triangleleft between trees: For $x, y \in Z$, $t \in T_x$, $t' \in T_y$, set $t \triangleright t'$ iff for some $Y \subset Y, y \rangle$ is a child of the root $\langle X, x \rangle$ in t , and t' is the subtree of t beginning at this $\langle Y, y \rangle$.

(C) The structure \mathcal{Z} : Let $\mathcal{Z} := \langle \langle x, t_x \rangle : x \in Z, t_x \in T_x \rangle, \langle x, t_x \rangle \succ \langle y, t_y \rangle$ iff $t_x \triangleright^* t_y \rangle$.

The rest of the proof are simple observations.

Fact 3.23

- (1) If $t_{U,x}$ is an U, x -tree, $\langle U_n, x_n \rangle$ an element of $t_{U,x}$, $\langle U_m, x_m \rangle$ a direct or indirect child of $\langle U_n, x_n \rangle$, then $x_m \notin H(U_n)$.
- (2) Let $\langle Y_n, y_n \rangle$ be an element in $t_{U,x} \in T\mu_x$, t' the subtree starting at $\langle Y_n, y_n \rangle$, then t' is a Y_n, y_n -tree.
- (3) \prec is free from cycles.
- (4) If $t_{U,x}$ is an U, x -tree, then $\langle x, t_{U,x} \rangle$ is \prec -minimal in $\mathcal{Z}[U]$.
- (5) No $\langle x, t_x \rangle$, $t_x \in T'_x$ is minimal in any $\mathcal{Z}[U]$, $U \in \mathcal{Y}$.
- (6) Smoothness is respected for the elements of the form $\langle x, t_{U,x} \rangle$.
- (7) Smoothness is respected for the elements of the form $\langle x, t_x \rangle$ with $t_x \in T'_x$.
- (8) $\mu = \mu_{\mathcal{Z}}$.

Proof:

- (1) trivial by (a) and (b).
- (2) trivial by (a).
- (3) Note that no $\langle x, t_x \rangle$, $t_x \in T'_x$ can be smaller than any other element (smaller elements require $U \neq \emptyset$ at the root). So no cycle involves any such $\langle x, t_x \rangle$. Consider now $\langle x, t_{U,x} \rangle$, $t_{U,x} \in T\mu_x$. For any $\langle y, t_{V,y} \rangle \prec \langle x, t_{U,x} \rangle$, $y \notin H(U)$ by (1), but $x \in \mu(U) \subseteq H(U)$, so $x \neq y$.
- (4) This is trivial by (1).
- (5) Let $x \in U \in \mathcal{Y}$, then f as used in the construction of level 1 of t_x chooses $y \in \mu(U) \neq \emptyset$, and some $\langle y, t_{U,y} \rangle$ is in $\mathcal{Z}[U]$ and below $\langle x, t_x \rangle$.
- (6) Let $x \in A \in \mathcal{Y}$, we have to show that either $\langle x, t_{U,x} \rangle$ is minimal in $\mathcal{Z}[A]$, or that there is $\langle y, t_y \rangle \prec \langle x, t_{U,x} \rangle$ minimal in $\mathcal{Z}[A]$. Case 1, $A \subseteq H(U)$: Then $\langle x, t_{U,x} \rangle$ is minimal in $\mathcal{Z}[A]$, again by (1). Case 2, $A \not\subseteq H(U)$: Then A is one of the Y_1 considered for level 1. So there is $\langle U \cup A, f_1(A) \rangle$ in level 1 with $f_1(A) \in \mu(A) \subseteq A$ by Fact 3.15, (3). But note that by (1) all elements below $\langle U \cup A, f_1(A) \rangle$ avoid $H(U \cup A)$. Let t be the subtree of $t_{U,x}$ beginning at $\langle U \cup A, f_1(A) \rangle$, then by (2) t is one of the $U \cup A, f_1(A)$ -trees, and $\langle f_1(A), t \rangle$ is minimal in $\mathcal{Z}[U \cup A]$ by (4), so in $\mathcal{Z}[A]$, and $\langle f_1(A), t \rangle \prec \langle x, t_{U,x} \rangle$.
- (7) Let $x \in A \in \mathcal{Y}$, $\langle x, t_x \rangle$, $t_x \in T'_x$, and consider the subtree t beginning at $\langle A, f(A) \rangle$, then t is one of the $A, f(A)$ -trees, and $\langle f(A), t \rangle$ is minimal in $\mathcal{Z}[A]$ by (4).
- (8) Let $x \in \mu(U)$. Then any $\langle x, t_{U,x} \rangle$ is \prec -minimal in $\mathcal{Z}[U]$ by (4), so $x \in \mu_{\mathcal{Z}}(U)$. Conversely, let $x \in U - \mu(U)$. By (5), no $\langle x, t_x \rangle$ is minimal in U . Consider now some $\langle x, t_{V,x} \rangle \in \mathcal{Z}$, so $x \in \mu(V)$. As $x \in U - \mu(U)$, $U \not\subseteq H(V)$ by Fact 3.15, (6). Thus U was considered in the construction of level 1 of $t_{V,x}$. Let t be the subtree of $t_{V,x}$ beginning at $\langle V \cup U, f_1(U) \rangle$, by $\mu(V \cup U) - H(V) \subseteq \mu(U)$ (Fact 3.15, (3)), $f_1(U) \in \mu(U) \subseteq U$, and $\langle f_1(U), t \rangle \prec \langle x, t_{V,x} \rangle$.

□

Proof of Proposition 3.21

Consider the construction in the proof of Proposition 3.22.

Thus, we only have to show that in \mathcal{Z} defined by

$$\mathcal{Z} := \langle \{ \langle x, t_x \rangle : x \in Z, t_x \in T_x \}, \langle x, t_x \rangle \succ \langle y, t_y \rangle \text{ iff } t_x \triangleright^* t_y, \langle x, t_x \rangle \triangleright \langle y, t_y \rangle \text{ implies } rg(y) \leq rg(x) \rangle.$$

But by construction of the trees, $x_n \in Y_{n+1}$, and $x_{n+1} \in \mu(U_n \cup Y_{n+1})$, so $rg(x_{n+1}) \leq rg(x_n)$.

□ (Proposition 3.21)

3.5 The logical properties with definability preservation

First, a small fact about the \mathcal{A} .

Fact 3.24

Let \mathcal{A} be as above (and thus finite). Then each A_i is equivalent to a formula α_i .

Proof:

We use the standard topology and its compactness. By definition, each $M(A_i)$ is closed, by finiteness all unions of such $M(A_i)$ are closed, too, so $\mathbf{C}(M(A_i))$ is closed. By compactness, each open cover $X_j : j \in J$ of the clopen $M(A_i)$ contains a finite subcover, so also $\bigcup \{M(A_j) : j \neq i\}$ has a finite open cover. But the $M(\phi)$, ϕ a formula form a basis of the closed sets, so we are done. □

Proposition 3.25

Let \sim be a logic for \mathcal{L} . Set $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, where \mathcal{M} is a preferential structure.

(1) Then there is a (transitive) definability preserving classical preferential model \mathcal{M} s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff

(LLE), (CCL), (SC), (PR) hold for all $T, T' \subseteq \mathcal{L}$.

(2) The structure can be chosen smooth, iff, in addition

(CUM) holds.

(3) The structure can be chosen \mathcal{A} -ranked, iff, in addition

(\mathcal{A} -min) $T \not\vdash \neg\alpha_i$ and $T \not\vdash \neg\alpha_j$, $i < j$ implies $\overline{\overline{T}} \vdash \neg\alpha_j$

holds.

The proof is an immediate consequence of Proposition 3.26 and the respective above results. This proposition (or its analogue) was mostly already shown in [Sch92] and [Sch96-1] and is repeated here for completeness' sake, but with a new and partly stronger proof.

Proposition 3.26

Consider for a logic \vdash on \mathcal{L} the properties

(LLE), (CCL), (SC), (PR), (CUM), (\mathcal{A} -min) hold for all $T, T' \subseteq \mathcal{L}$.

and for a function $\mu : \mathcal{D}_{\mathcal{L}} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ the properties

(μdp) μ is definability preserving, i.e. $\mu(M(T)) = M(T')$ for some T'

($\mu \subseteq$), (μPR), (μCUM), ($\mu \mathcal{A}$)

for all $X, Y \in \mathcal{D}_{\mathcal{L}}$.

It then holds:

(a) If μ satisfies (μdp), ($\mu \subseteq$), (μPR), then \vdash defined by $\overline{\overline{T}} := T^\mu := Th(\mu(M(T)))$ satisfies (LLE), (CCL), (SC), (PR). If μ satisfies in addition (μCUM), then (CUM) will hold, too. If μ satisfies in addition ($\mu \mathcal{A}$), then (\mathcal{A} -min) will hold, too.

(b) If \vdash satisfies (LLE), (CCL), (SC), (PR), then there is $\mu : \mathcal{D}_{\mathcal{L}} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ s.t. $\overline{\overline{T}} = T^\mu$ for all $T \subseteq \mathcal{L}$ and μ satisfies (μdp), ($\mu \subseteq$), (μPR). If, in addition, (CUM) holds, then (μCUM) will hold, too. If, in addition, (\mathcal{A} -min) holds, then ($\mu \mathcal{A}$) will hold, too.

Proof:

Set $\mu(T) := \mu(M(T))$, note that $\mu(T \cup T') := \mu(M(T \cup T')) = \mu(M(T) \cap M(T'))$.

(a) Suppose $\overline{\overline{T}} = T^\mu$ for some such μ , and all T .

(LLE): If $\overline{\overline{T}} = \overline{\overline{T'}}$, then $M(T) = M(T')$, so $\mu(T) = \mu(T')$, and $T^\mu = T'^\mu$.

(CCL) and (SC) are trivial.

We show (PR): $M(\overline{\overline{T}} \cup T') = M(\overline{\overline{T}}) \cap M(T') \stackrel{(\mu dp)}{=} \mu(T) \cap M(T') \stackrel{(\mu \subseteq)}{=} \mu(T) \cap M(T) \cap M(T') = \mu(T) \cap M(T \cup T') \stackrel{(\mu PR)}{\subseteq} \mu(T \cup T') \stackrel{(\mu dp)}{=} M(\overline{\overline{T \cup T'}})$. Let now $\phi \in \overline{\overline{T \cup T'}}$, so ϕ holds in all $m \in M(\overline{\overline{T \cup T'}})$, so ϕ holds in all $m \in M(\overline{\overline{T}} \cup T')$, so $\overline{\overline{T}} \cup T' \vdash \phi$, so $\phi \in \overline{\overline{T}} \cup T'$.

We turn to (CUM):

Let $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}}$. Thus by (μCum) and $\mu(T) \subseteq M(\overline{\overline{T}}) \subseteq M(T') \subseteq M(T)$, so $\mu(T) = \mu(T')$, so $\overline{\overline{T}} = Th(\mu(T)) = Th(\mu(T')) = \overline{\overline{T'}}$.

(\mathcal{A} -min) is trivial.

(b) Let \vdash satisfy (LLE) – (CUM) for all T . We define μ and show $\overline{\overline{T}} = T^\mu$. (CUM) will be needed only to show (μCUM).

If $X = M(T)$ for some $T \subseteq \mathcal{L}$, set $\mu(X) := M(\overline{\overline{T}})$.

If $X = M(T) = M(T')$, then $\overline{\overline{T}} = \overline{\overline{T'}}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by (LLE), so $M(\overline{\overline{T}}) = M(\overline{\overline{T'}})$, and μ is well-defined. Moreover, μ satisfies (μdp), and by (SC), $\mu(X) \subseteq X$. We show $\overline{\overline{T}} = T^\mu$: Let now $T \subseteq \mathcal{L}$ be given. Then $\phi \in T^\mu : \leftrightarrow \forall m \in \mu(T).m \models \phi \leftrightarrow \forall m \in M(\overline{\overline{T}}).m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed).

Next, we show that the above defined μ satisfies (μPR). Suppose $X := M(T)$, $Y := M(T')$, $X \subseteq Y$, we have to show $\mu(Y) \cap X \subseteq \mu(X)$. By prerequisite, $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$, so $\overline{\overline{T \cup T'}} = \overline{\overline{T}}$, so $\overline{\overline{T \cup T'}} = \overline{\overline{T}}$ by (LLE). By

(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T'}} \cup T$, so $\mu(Y) \cap X = \mu(T') \cap M(T) = M(\overline{\overline{T'}} \cup T) \subseteq M(\overline{\overline{T \cup T'}}) = M(\overline{\overline{T}}) = \mu(X)$, using (μdp).

$(\mu \mathcal{A})$ is trivial.

It remains to show (μCUM) . So let $X = M(T)$, $Y = M(T')$, and $\mu(T) := M(\overline{\overline{T}}) \subseteq M(T') \subseteq M(T) \rightarrow \overline{\overline{T}} \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} = \overline{\overline{\overline{\overline{T}}}} \rightarrow \overline{\overline{T}} = \overline{\overline{\overline{\overline{T}}}} = \overline{\overline{\overline{\overline{T'}}}} = \overline{\overline{T'}} \rightarrow \mu(T) = M(\overline{\overline{T}}) = M(\overline{\overline{T'}}) = \mu(T')$, thus $\mu(X) = \mu(Y)$.

□ (Proposition 3.26)

4 Formal results and representation for hierarchical conditionals

We look here at the following problem:

Given

(1.1) a finite, ordered partition \mathcal{A} of \mathbf{A} , $\mathcal{A} = \langle \{A_i : i \in I\}, < \rangle$

(1.2) a normality relation \prec , which is an \mathcal{A} -ranking, defining a choice function μ on subsets of \mathbf{A} , (so, obviously, $A < A'$ iff $\mu(A \cup A') \cap A' = \emptyset$),

(1.3) a subset $\mathbf{B} \subseteq \mathbf{A}$, and we set $\mathcal{C} := \langle \mathcal{A}, \mathbf{B} \rangle$ (thus, the B_i are just $A_i \cap \mathbf{B}$, this way of writing saves a little notation),

(2.1) a set of models M ,

(2.2) an accessibility relation R on M , with some finite upper bound on R -chains,

(2.3) an unknown extension of R to pairs (m, a) , $m \in M$, $a \in \mathbf{A}$,

(3.1) a notion of validity $m \models \mathcal{C}$, for $m \in M$, defined by $m \models \mathcal{C}$ iff $\{a \in \mathbf{A} : \exists A \in \mathcal{A} (a \in \mu(A), a \in R(m))\}$, and

$$\neg \exists a' (\exists A' \in \mathcal{A} (a' \in \mu(A'), a' \in R(m), a' \prec a) \subseteq \mathbf{B},$$

(3.2) a subset M' of M

give a criterion which decides whether it is possible to construct the extension of R to pairs (m, a) s.t. $\forall m \in M. (m \in M' \Leftrightarrow m \models \mathcal{C})$.

We first show some elementary facts on the situation, and give the criterion in Proposition 4.4, together with the proof that it does what is wanted.

Fact 4.1

Reachability for a transitive relation is characterized by

$$y \in R(x) \rightarrow R(y) \subseteq R(x)$$

Proof:

Define directly xRz iff $z \in R(x)$. This does it. \square

Let now S be a set with an accessibility relation R' , generated by transitive closure from the intransitive subrelation R . All modal notation will be relative to this R .

$$m' \not\models C$$

Let $\mathbf{A} = M(\alpha)$, $A_i = M(\alpha_i)$, the latter is justified by Fact 3.24.

Definition 4.1

$$(1) \mu(\mathcal{A}) := \bigcup \{ \mu(A_i) : i \in I \}$$

(warning: this is NOT $\mu(\mathbf{A})$)

$$(2) \mathcal{A}_m := R(m) \cap \mathbf{A},$$

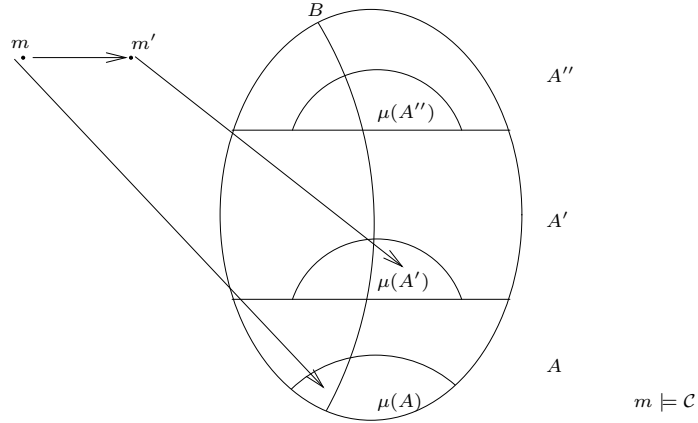
$$(3) \mu(\mathcal{A}_m) := \bigcup \{ \mu(A_i) \cap R(m) : i \in I \}$$

$$(3a) \nu(\mathcal{A}_m) := \mu(\mu(\mathcal{A}_m))$$

(thus $\nu(\mathcal{A}_m) = \{a \in \mathbf{A} : \exists A \in \mathcal{A}(a \in \mu(A), a \in R(m), \text{ and } \neg \exists a'(\exists A' \in \mathcal{A}(a' \in \mu(A'), a' \in R(m), a' \prec a))\}$).

$$(4) m \models C :\leftrightarrow \nu(\mathcal{A}_m) \subseteq B.$$

See Diagram 4.1



Here, the “best” element m sees is in B , so C holds in m .

The “best” element m' sees is not in B , so C does not hold in m' .

Diagram 4.1 Validity of C from m and m'

We have the following Fact for $m \models C$:

Fact 4.2

Let $m, m' \in M$, $A, A' \in \mathcal{A}$.

- (1) $m \models \Box \neg \alpha, mRm' \Rightarrow m' \models \Box \neg \alpha$
- (2) $mRm', \nu(\mathcal{A}_m) \cap A \neq \emptyset, \nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset, \Rightarrow A \leq A'$
- (3) $mRm', \nu(\mathcal{A}_m) \cap A \neq \emptyset, \nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset, m \models \mathcal{C}, m' \not\models \mathcal{C}, \Rightarrow A < A'$

Proof:

Trivial. \square

Fact 4.3

We can conclude from above properties that there are no arbitrarily long R -chains of models m , changing from $m \models \mathcal{C}$ to $m \not\models \mathcal{C}$ and back.

Proof:

Trivial: By Fact 4.2, (3), any change from $\models \mathcal{C}$ to $\not\models \mathcal{C}$ results in a strict increase in rank. \square

We solve now the representation task described at the beginning of Section 4, all we need are the properties shown in Fact 4.2.

(Note that constructing R between the different m, m' is trivial: we could just choose the empty relation.)

Proposition 4.4

If the properties of Fact 4.2 hold, we can extend R to solve the representation problem described at the beginning of this Section 4.

Proof:

By induction on R . This is possible, as the depth of R on M was assumed to be finite.

Construction 4.1

We choose now elements as possible, which ones are chosen exactly does not matter.

$X_i := \{b_i, c_i\}$ iff $\mu(A_i) \cap \mathbf{B} \neq \emptyset$ and $\mu(A_i) - \mathbf{B} \neq \emptyset$, $b_i \in \mu(A_i) \cap \mathbf{B}$, $c_i \in \mu(A_i) - \mathbf{B}$.

$X_i := \{c_i\}$ iff $\mu(A_i) \cap \mathbf{B} = \emptyset$ and $\mu(A_i) - \mathbf{B} \neq \emptyset$, $c_i \in \mu(A_i) - \mathbf{B}$

$X_i := \{b_i\}$ iff $\mu(A_i) \cap \mathbf{B} \neq \emptyset$ and $\mu(A_i) - \mathbf{B} = \emptyset$, $b_i \in \mu(A_i) \cap \mathbf{B}$,

$X_i := \emptyset$ iff $\mu(A_i) = \emptyset$.

Case 1:

Let m be R -minimal and $m \models \mathcal{C}$. Let i_0 be the first i s.t. $b_i \in X_i$, make $\gamma(m) := i_0$, and make $R(m) := \{b_{i_0}\} \cup \bigcup \{X_i : i > i_0\}$. This makes \mathcal{C} hold. (This leaves us as many possibilities open as possible - remember we have to decrease the set of reachable elements now.)

Case 2:

Let m be R -minimal and $m \not\models \mathcal{C}$. Let i_0 be the first i s.t. $c_i \in X_i$, make $\gamma(m) := i_0$, and make $R(m) := \bigcup \{X_i : i \geq i_0\}$. This makes \mathcal{C} false.

Let all R -predecessors of m be determined, and $i := \max\{\gamma(m') : m' R m\}$.

Case 3: $m \models \mathcal{C}$. Let j be the smallest $i' \geq i$ with $\mu(A_{i'}) \cap \mathbf{B} \neq \emptyset$. Let $R(m) := \{b_j\} \cup \bigcup \{X_k : k > j\}$, and $\gamma(m) := j$.

Case 4: $m \not\models \mathcal{C}$.

Case 4.1: For all $m' R m$ with $i = \gamma(m')$ $m' \not\models \mathcal{C}$.

Take one such m' and set $R(m) := R(m')$, $\gamma(m) := i$.

Case 4.2: There is $m' R m$ with $i = \beta(m')$ $m' \models \mathcal{C}$.

Let j be the smallest $i' > i$ with $\mu(A_{i'}) - \mathbf{B} \neq \emptyset$. Let $R(m) := \bigcup \{X_k : k \geq j\}$. (Remark: To go from \models to $\not\models$, we have to go higher in the hierarchy.)

Obviously, validity is done as it should be. It remains to show that the sets of reachable elements decrease with R .

Fact 4.5

In above construction, if $m R m'$, then $R(m') \subseteq R(m)$.

Proof:

By induction, considering R . \square (Fact 4.5 and Proposition 4.4)

We consider an example for illustration.

Example 4.1

Let $a_1 R a_2 R c_1$, $b_1 R b_2 R b_3 R c R d_1 R d_2$.

Let $\mathcal{C} = (A_1 > B_1, \dots, A_n > B_n)$ with the C_i consistency with $\mu(A_i)$.

Let $\mu(A_2) \cap B_2 = \emptyset$, $\mu(A_3) \subseteq B_3$, and for the other i hold neither of these two.

Let $a_1, a_2, b_2, c_1, d_2 \models \mathcal{C}$, the others $\not\models \mathcal{C}$.

Let $\mu(A_1) = \{a_{1,1}, a_{1,2}\}$, with $a_{1,1} \in B_1$, $a_{1,2} \notin B_1$,

$\mu(A_2) = \{a_{2,1}\}$, $\mu(A_3) = \{a_{3,1}\}$ (there is no reason to differentiate),

and the others like $\mu(A_i)$. Let $\mu A := \bigcup \{\mu(A_i) : i \leq n\}$.

We have to start at a_1 and b_1 , and make $R(x)$ progressively smaller.

Let $R(a_1) := \mu A - \{a_{1,2}\}$, so $a_1 \models \mathcal{C}$. Let $R(a_2) = R(a_1)$, so again $a_2 \models \mathcal{C}$.

Let $R(b_1) := \mu A - \{a_{1,1}\}$, so $b_1 \not\models \mathcal{C}$. We now have to take $a_{1,2}$ away, but $a_{2,1}$ too to be able to change. So let $R(b_2) := R(b_1) - \{a_{1,2}, a_{2,1}\}$, so we begin at $\mu(A_3)$, which is a (positive) singleton. Then let $R(b_3) := R(b_2) - \{a_{3,1}\}$.

We can choose $R(c) := R(b_3)$, as $R(b_3) \subseteq R(a_2)$.

Let $R(c_1) := R(c) - \{a_{4,2}\}$ to make \mathcal{C} hold again. Let $R(d_1) := R(c)$, and $R(d_2) := R(c_1)$.

\square

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